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A HYBRID BLOCK METHOD FOR DIRECT SOLUTION OF GENERAL FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
Many problems in applied sciences and engineering, such as static deflection of a uniform beam, fluid dynamics, neural networks, electric circuits, and the illposed problem of a beam on elastic foundation often result to fourth-order initial value problem of ordinary differential equations (ODEs). In this study a direct solution for the general fourth-order initial value problems of (ODEs) was derived using Linear Multi-step Method. Power series and its derivatives are adopted as basis function and differential for the purpose of interpolation and collocation respectively. This interpolation and collocation are carried out at selected nodal and off-nodal points to generate a set of linear equations. Solving these equations give the values for the unknown coefficient parameters to be used to determine the required continuous method after necessary simplification. The additional methods are obtained for the implementation of the main methods in block mode. The basic properties of the hybrid methods is established to confirm their usability, accuracy and efficiency. Accuracy and efficiency of the methods is determined by applying the derived methods to solve linear and non-linear test problems. The results obtained is compared with those of cited methods in literature.
Keywords: *Convergence, Hybrid, Block, Multistep, Off-grid points.*

Introduction

Numerous physical problems in sciences and engineering are usually modeled mathematically as a fourth-order ordinary differential equations (ODEs) with initial conditions according to [1,2]. Fourth-order ordinary differential equations have gained significant attention due to their applications in structural mechanics, fluid dynamics, and quantum mechanics as discussed in the studies of [3,4, [5]]. Scholars have developed various numerical approaches to address fourth-order ordinary differential equations (ODEs), among which the linear multistep method (LMM) has been widely examined, as noted by [5]. According to [6], LMMs constitute a class of numerical techniques that estimate the solution of a differential equation through a linear combination of prior function values and their derivatives. These methods present notable benefits, such as achieving higher-order accuracy compared to single-step approaches like Runge-Kutta, as highlighted in [7]. This characteristic renders them particularly efficient for solving higher-order differential equations, including fourth-order ODEs, as explored in [8].

The present study focuses on deriving numerical solutions for fourthorder problems incorporating lowerorder derivatives, expressed as:

$$y^4 = f(x, y, y', y'', y''') \,, \quad y^r(x_0) = y_0^r, r = 0,1,2,3 \qquad (1)$$

It is assumed that f in (1) is a continuous real-value function [28]. The reduction approach was the first to be adopted for the equation of form (1) due to the availability of methods to handle the equivalent version of its first-order system, as reported in the work of [5] and several others. As time passed, authors such as [7-13] opined that due to the complexity encountered in the reduction approach and the increase in the scale of differential equations, the approach was considered not suitable for a larger system; hence the need for the direct approach proposed in the studies of [8,14,15] and many others.

A comprehensive review of the literature $[3,16-36]$ reveals that researchers have independently developed various numerical methods for solving fourth-order ordinary differential equations (ODDEs). A common characteristic of these approaches is their formulation as implicit linear multistep methods (LMMs) implemented in block form. While many of these existing methods possess an order of accuracy equal to or higher than the one proposed in this study, their computational performance does not surpass the present method. This discrepancy may be attributed to the hybrid nature of the current approach, which likely enhances its efficiency and effectiveness.

More so, most of the methods mentioned above for solving higher order ODEs which were implemented in block mode was an attempt to overcome very early setback of predictor-corrector method for instance, the combination of predictors of lower order with the correctors in the predictor-corrector method and they are more or less have low order of accuracy. In the implementation, it should be noted that the block method is problem independent as against the conventional block methods of problem dependent and hence the motivation of this work. In this paper, an order seven block method with four inter-steps embedded in the step length of three is presented for the solution of general fourth order ODEs.

In particular, [28] introduced a hybrid linear multistep method specifically designed to solve equations of the form (1) using power series polynomial as basis function. While this approach provided a direct numerical solution to the problem, its error performance indicates room for further refinement. Consequently, this study investigates the development and implementation of an optimized numerical algorithm for the direct solution of fourth-order ordinary differential equations, with the dual objectives of enhancing computational efficiency and improving upon the accuracy limitations of current methods.

Methodology

Let the power series in equation (2) be the approximate solution of equation (1)

$$y(x) = \sum_{j=0}^{(C+I)-1} e_j x^j \tag{2}$$

where

$e_j, j = 0,1,2,\cdots,k$ are the coefficients to be determined, x is continuous and differentiable C is the number of collocation points and I is the number of interpolation points. The fourth derivatives of (3) is gotten as

$$y^{iv}(x) = \sum_{j=4}^{(C+I)-1} j(j-1)(j-2)(j-3)e_j x^{j-4} \tag{3}$$

Equations (3) and (4) are respectively the interpolation and collocation equations which would be evaluated at selected grid and off-grid points. The interpolation equation is given in expanded form below,

$$y(x) = e_0 + e_1x + e_2x^2 + e_3x^3 + e_4x^4 + e_5x^5 + e_6x^6 + e_7x^7 + e_8x^8 + e_9x^9 + e_{10}x^{10} \tag{4}$$

while the collocation equations is given in expanded form as follow,

$y^{iv}(x) = 5040e_{10}x^6 + 3024e_9x^5 + 1680e_8x^4 + 840e_7x^3 + 360e_6x^2 + 120e_5x + 24e_4$ (5)
Interpolating (5) at points $x = x = x_n, x_{n+r}, x_{n+s}, x_{n+1}, x_{n+2}, x_{n+u}, x_{n+v}$, and collocating equation (6) at the points $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}$.
Where $0 < r, s < 1$ and $2 < u, v < 3$.

Specifically, the values of r, s, p, u, v are taken to be $\frac{1}{3}, \frac{2}{3}, \frac{7}{3}, \frac{8}{3}$, respectively. Gaussian elimination approach is employed to solve the resulting equations arising from the interpolation and collocation of equations (5) and (6) to determine the values of $e_j's, j = 0(1)10$ taking $x_n = 0$. The values of $e_j, j = 0,1, \dots, 10$ are substituting into (5) to gives a linear hybrid multi-step method with continuous coefficients in the form:

$$y(x) = \begin{Bmatrix} \alpha_0(t)y_n + \alpha_{\frac{1}{3}}(t)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(t)y_{n+\frac{2}{3}} + \alpha_1(t)y_{n+1} + \alpha_2(t)y_{n+2} + \alpha_{\frac{7}{3}}(t)y_{n+\frac{7}{3}} \\ +\alpha_{\frac{8}{3}}(t)y_{n+\frac{8}{3}} + h^4(\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3} \end{Bmatrix} \quad (6)$$

Using the transformation in Ref. [23],

$$t = \frac{x - x_{n+k-1}}{h}$$

$$\frac{dt}{dx} = \frac{1}{h}$$

The coefficients of y_{n+j} and f_{n+j} are obtained in term of t as:

$$\alpha_0(t)=\left\{\begin{array}{c}1-\frac{593443869}{86198392}t+\frac{359534717}{21549598}t^2-\frac{801183393}{49256224}t^3+\frac{45937935}{3078514}t^5-\frac{190414557}{12314056}t^6\\+\frac{1383087231}{172396784}t^7-\frac{816428241}{344793568}t^8+\frac{18600435}{49256224}t^9-\frac{1240029}{49256224}t^{10}\end{array}\right\} \quad (7)$$

$$\alpha_{\frac{1}{3}}(t)=\left\{\begin{array}{c}\frac{41608474752}{2824536595}t-\frac{1083789646416}{19771756165}t^2+\frac{254754026400}{3954351233}t^3-\frac{206901371196}{2824536595}t^5\\+\frac{237901406562}{2824536595}t^6-\frac{137385640701}{2824536595}t^7+\frac{1257519744639}{79087024660}t^8-\frac{21943301679}{7908702466}t^9\\+\frac{2250140877}{11298146380}t^{10}\end{array}\right\} \quad (8)$$

$$\alpha_{\frac{2}{3}}(t)=\left\{\begin{array}{c}-\frac{7618699515}{564907319}t+\frac{315396476253}{4519258552}t^2-\frac{887705884113}{9038517104}t^3+\frac{789119229591}{5649073190}t^5\\-\frac{495277314021}{2824536595}t^6+\frac{2460141566181}{22596292760}t^7-\frac{1701251538291}{45192585520}t^8+\frac{61876382031}{9038517104}t^9\\-\frac{22869697383}{45192585520}t^{10}\end{array}\right\} \quad (9)$$

$$\alpha_1(t)=\left\{\begin{array}{c}+\frac{18835166336}{2824536595}t-\frac{107351964112}{2824536595}t^2+\frac{34788017436}{564907319}t^3-\frac{60198708087}{564907319}t^5\\+\frac{804974407083}{5649073190}t^6-\frac{261083902536}{2824536595}t^7+\frac{743360958099}{22596292760}t^8-\frac{3443335272}{564907319}t^9\\+\frac{2060869149}{4519258552}t^{10}\end{array}\right\} \quad (10)$$

$$\alpha_2(t)=\left\{\begin{array}{c}-\frac{9794493107}{2824536595}t+\frac{490116950269}{22596292760}t^2-\frac{362207522031}{9038517104}t^3+\frac{512648854767}{5649073190}t^5\\-\frac{737408741139}{5649073190}t^6+\frac{1989446210163}{22596292760}t^7+\frac{53858118321}{9038517104}t^9\\-\frac{20196942813}{45192585520}t^{10}\end{array}\right\} \quad (11)$$

$$\alpha_{\frac{7}{3}}(t)=\left\{\begin{array}{c}\frac{13731589632}{3954351233}t-\frac{86400143712}{3954351233}t^2+\frac{22980109524}{564907319}t^3-\frac{263913685893}{2824536595}t^5\\+\frac{761474947437}{5649073190}t^6-\frac{1796423641569}{19771756165}t^7+\frac{5196403641861}{158174049320}t^8-\frac{6908341527}{1129814638}t^9\\+\frac{10333240389}{22596292760}t^{10}\end{array}\right\} \quad (12)$$

$$\alpha_{\frac{8}{3}}(t)=\left\{\begin{array}{c}-\frac{23344358343}{22596292760}t+\frac{514676383281}{79087024660}t^2-\frac{1535161461207}{126539239456}t^3+\frac{31510599993}{1129814638}t^5\\-\frac{907744481757}{22596292760}t^6+\frac{1219881881421}{45192585520}t^7-\frac{6150732689637}{632696197280}t^8+\frac{228066656373}{126539239456}t^9\\-\frac{2428547589}{18077034208}t^{10}\end{array}\right\} \quad (13)$$

$$\beta_0=\left\{\begin{array}{c}-\frac{6180594428}{6177261533265}th^4+\frac{45859834369}{5765444097714}t^2h^4-\frac{4321725420727}{172963322931420}t^3h^4+\frac{1}{24}h^4t^4\\-\frac{16836604981}{406733269680}t^5h^4+\frac{5243265089}{203366634840}t^6h^4-\frac{827987401}{81346653936}t^7h^4+\frac{783034047}{316348098640}t^8h^4\\-\frac{32096933}{94904429592}t^9h^4+\frac{896787}{45192585520}t^{10}h^4\end{array}\right\} \quad (14)$$

$$\beta_1(t)=\left\{\begin{array}{c}\frac{19893911528}{2059087177755}th^4-\frac{250818744247}{4804536748095}t^2h^4+\frac{2161991632091}{28827220488570}t^3h^4-\frac{1069014199}{22596292760}t^5h^4\\-\frac{8033333893}{406733269680}t^6h^4+\frac{1745826067}{27115551312}t^7h^4-\frac{12098711521}{316348098640}t^8h^4+\frac{587147731}{63269619728}t^9h^4\\-\frac{9207117}{11298146380}t^{10}h^4\end{array}\right. \quad (15)$$

$$\beta_2 = \begin{cases} \frac{33917903872}{2059087177755} th^4 - \frac{498565998146}{4804536748095} t^2 h^4 \frac{11150475106553}{57654440977140} t^3 h^4 - \frac{20016159647}{45192585520} t^5 h^4 \\ + \frac{32274159379}{50841658710} t^6 h^4 - \frac{11478928105}{27115551312} t^7 h^4 + \frac{47763496271}{316348098640} t^8 h^4 - \frac{875841569}{31634809864} t^9 h^4 \\ + \frac{18456507}{9038517104} t^{10} h^4 \end{cases} \quad (16)$$

$$\beta_3(t) = \begin{cases} \frac{599647448}{6177261533265} \\ + \frac{1608095383}{406733269680} t^6 h^4 - \frac{222188237}{81346653936} t^7 h^4 + \frac{323588393}{316348098640} t^8 h^4 - \frac{37977833}{189808859184} t^9 h^4 \\ + \frac{362493}{22596292760} t^{10} h^4 \end{cases} \quad (17)$$

The first derivative gives:

$$\alpha_0'(t) = \begin{cases} -\frac{593443869}{86198392} + \frac{359534717}{10774799} t - \frac{2403550179}{49256224} t^2 + \frac{229689675}{3078514} t^4 - \frac{571243671}{6157028} t^5 \\ + \frac{1383087231}{24628112} t^6 - \frac{816428241}{43099196} t^7 + \frac{167403915}{49256224} t^8 - \frac{6200145}{24628112} t^9 \end{cases}$$

$$\alpha_{\frac{1}{3}}'(t) = \begin{cases} \frac{41608474752}{2824536595} - \frac{2167579292832}{19771756165} t + \frac{764262079200}{3954351233} t^2 - \frac{206901371196}{564907319} t^4 \\ + \frac{1427408439372}{2824536595} t^5 - \frac{961699484907}{2824536595} t^6 + \frac{2515039489278}{19771756165} t^7 \\ - \frac{197489715111}{7908702466} t^8 + \frac{2250140877}{1129814638} t^9 \end{cases}$$

$$\alpha_{\frac{2}{3}}'(t) = \begin{cases} -\frac{7618699515}{564907319} + \frac{315396476253}{2259629276} t - \frac{2663117652339}{9038517104} t^2 + \frac{789119229591}{1129814638} t^4 \\ + \frac{556887438279}{9038517104} t^8 - \frac{22869697383}{4519258552} t^9 \\ \frac{18835166336}{2824536595} - \frac{214703928224}{2824536595} t + \frac{104364052308}{564907319} t^2 - \frac{300993540435}{564907319} t^4 \\ + \frac{2414923221249}{2824536595} t^5 - \frac{1827587317752}{2824536595} t^6 + \frac{743360958099}{2824536595} t^7 \\ - \frac{30990017448}{564907319} t^8 + \frac{10304345745}{2259629276} t^9 \end{cases}$$

$$\alpha_1'(t) = \begin{cases} -\frac{9794493107}{2824536595} + \frac{490116950269}{11298146380} t - \frac{1086622566093}{9038517104} t^2 + \frac{512648854767}{1129814638} t^4 \\ - \frac{2212226223417}{2824536595} t^5 + \frac{13926123471141}{22596292760} t^6 - \frac{1442391378813}{5649073190} t^7 \\ + \frac{484723064889}{9038517104} t^8 - \frac{20196942813}{4519258552} t^9 \end{cases}$$

$$\alpha'_{\frac{7}{3}}(t)=\begin{Bmatrix}\frac{13731589632}{3954351233}-\frac{172800287424}{3954351233}t+\frac{68940328572}{564907319}t^2-\frac{263913685893}{564907319}t^4\\+\frac{2284424842311}{2824536595}t^5-\frac{1796423641569}{2824536595}t^6+\frac{5196403641861}{19771756165}t^7\\-\frac{62175073743}{1129814638}t^8+\frac{10333240389}{2259629276}t^9\end{Bmatrix} \quad (23)$$

$$\alpha'_{\frac{8}{3}}(t)=\begin{Bmatrix}-\frac{23344358343}{22596292760}+\frac{514676383281}{39543512330}t-\frac{4605484383621}{126539239456}t^2+\frac{157552999965}{1129814638}t^4\\-\frac{2723233445271}{11298146380}t^5+\frac{8539173169947}{45192585520}t^6-\frac{6150732689637}{79087024660}t^7\\+\frac{2052599907357}{126539239456}t^8-\frac{12142737945}{9038517104}t^9\end{Bmatrix} \quad (24)$$

$$\beta'_0(t)=\begin{Bmatrix}-\frac{6180594428}{6177261533265}h^4+\frac{45859834369}{2882722048857}th^4-\frac{4321725420727}{57654440977140}t^2h^4+\frac{1}{6}t^3h^4\\-\frac{16836604981}{81346653936}h^4t^4+\frac{5243265089}{33894439140}t^5h^4-\frac{5795911807}{81346653936}t^6h^4+\frac{783034047}{39543512330}t^7h^4\\-\frac{96290799}{31634809864}t^8h^4+\frac{896787}{4519258552}t^9h^4\end{Bmatrix} \quad (25)$$

$$\beta'_1(t)=\begin{Bmatrix}\frac{19893911528}{2059087177755}h^4-\frac{501637488494}{4804536748095}th^4+\frac{2161991632091}{9609073496190}t^2h^4-\frac{1069014199}{4519258552}h^4t^4\\-\frac{8033333893}{67788878280}t^5h^4+\frac{12220782469}{27115551312}t^6h^4-\frac{12098711521}{39543512330}t^7h^4\frac{5284329579}{63269619728}t^8h^4\\-\frac{9207117}{1129814638}t^9h^4\end{Bmatrix} \quad (26)$$

$$\left.\begin{matrix}\frac{33917903872}{2059087177755}h^4-\frac{997131996292}{4804536748095}th^4+\frac{11150475106553}{19218146992380}t^2h^4-\frac{20016159647}{9038517104}h^4t^4\\+\frac{32274159379}{8473609785}t^5h^4+\frac{47763496271}{39543512330}t^7h^4-\frac{7882574121}{31634809864}t^8h^4+\frac{92282535}{4519258552}t^9h^4\end{matrix}\right\} \quad (27)$$

$$\beta'_2(t)=\begin{Bmatrix}\frac{599647448}{6177261533265}h^4-\frac{17675364002}{14413610244285}th^4+\frac{99316143581}{28827220488570}t^2h^4-\frac{547765327}{40673326968}h^4t^4\\+\frac{1608095383}{67788878280}t^5h^4-\frac{1555317659}{81346653936}t^6h^4+\frac{323588393}{39543512330}t^7h^4-\frac{1139333499}{63269619728}t^8h^4\\+\frac{362493}{2259629276}t^9h^4\end{Bmatrix} \quad (28)$$

The second derivatives give:

$$\alpha''_0(t)=\begin{Bmatrix}\frac{359534717}{10774799}-\frac{2403550179}{24628112}t+\frac{459379350}{1539257}t^3-\frac{2856218355}{6157028}t^4\\+\frac{4149261693}{12314056}t^5-\frac{816428241}{6157028}t^6+\frac{167403915}{6157028}t^7-\frac{55801305}{24628112}t^8\end{Bmatrix} \quad (29)$$

$$\alpha''_{\frac{1}{3}}(t)=\begin{Bmatrix}-\frac{2167579292832}{19771756165}+\frac{1528524158400}{3954351233}t-\frac{827605484784}{564907319}t^3+\frac{1427408439372}{564907319}t^4\\-\frac{5770196909442}{2824536595}t^5+\frac{2515039489278}{2824536595}t^6-\frac{789958860444}{3954351233}t^7+\frac{20251267893}{1129814638}t^8\end{Bmatrix} \quad (30)$$

$$\alpha''_{\frac{2}{3}}(t)=\begin{Bmatrix}\frac{315396476253}{2259629276}-\frac{2663117652339}{4519258552}t+\frac{1578238459182}{564907319}t^3-\frac{2971663884126}{564907319}t^4\\-\frac{11908760768037}{5649073190}t^6+\frac{556887438279}{1129814638}t^7-\frac{205827276447}{4519258552}t^8\end{Bmatrix} \quad (31)$$

$$\alpha''_1(t)=\begin{Bmatrix}-\frac{214703928224}{2824536595}+\frac{208728104616}{564907319}t-\frac{1203974161740}{564907319}t^3+\frac{2414923221249}{564907319}t^4\\-\frac{10965523906512}{2824536595}t^5+\frac{5203526706693}{2824536595}t^6-\frac{247920139584}{564907319}t^7+\frac{92739111705}{2259629276}t^8\end{Bmatrix} \quad (32)$$

$$\alpha_2''(t)=\begin{cases}\frac{490116950269}{11298146380}-\frac{1086622566093}{4519258552}t+\frac{1025297709534}{564907319}t^3-\frac{2212226223417}{564907319}t^4\\+\frac{41778370413423}{11298146380}t^5-\frac{10096739651691}{5649073190}t^6+\frac{484723064889}{1129814638}t^7-\frac{181772485317}{4519258552}t^8\end{cases}(33)$$

$$\alpha_{\frac{7}{3}}''(t)=\begin{cases}-\frac{172800287424}{3954351233}+\frac{137880657144}{564907319}t-\frac{1055654743572}{564907319}t^3+\frac{2284424842311}{564907319}t^4\\-\frac{10778541849414}{2824536595}t^5+\frac{5196403641861}{2824536595}t^6-\frac{248700294972}{564907319}t^7+\frac{92999163501}{2259629276}t^8\end{cases}(34)$$

$$\alpha_{\frac{8}{3}}''(t)=\begin{cases}\frac{514676383281}{39543512330}-\frac{4605484383621}{63269619728}t+\frac{315105999930}{564907319}t^3-\frac{2723233445271}{2259629276}t^4\\+\frac{25617519509841}{22596292760}t^5-\frac{6150732689637}{11298146380}t^6+\frac{2052599907357}{15817404932}t^7-\frac{109284641505}{9038517104}t^8\end{cases}(35)$$

$$\beta_0''(t)=\begin{cases}\frac{45859834369}{2882722048857}h^4-\frac{4321725420727}{28827220488570}th^4+\frac{1}{2}t^2h^4-\frac{16836604981}{20336663484}t^3h^4\\+\frac{5243265089}{6778887828}h^4t^4-\frac{5795911807}{13557775656}t^5h+\frac{783034047}{5649073190}t^6h^{44}-\frac{96290799}{3954351233}t^7h^4\\+\frac{8071083}{4519258552}t^8h^4\end{cases}(36)$$

$$\beta_1''(t)=\begin{cases}-\frac{501637488494}{4804536748095}h^4+\frac{2161991632091}{4804536748095}th^4-\frac{1069014199}{1129814638}t^3h^4-\frac{8033333893}{13557775656}h^4t^4\\+\frac{12220782469}{4519258552}t^5h^4-\frac{12098711521}{5649073190}t^6h^4+\frac{5284329579}{7908702466}t^7h^4-\frac{82864053}{1129814638}t^8h^4\end{cases}(37)$$

$$\beta_2''(t)=\begin{cases}-\frac{997131996292}{4804536748095}h^4+\frac{11150475106553}{9609073496190}th^4-\frac{20016159647}{2259629276}t^3h^4+\frac{32274159379}{1694721957}h^4t^4\\-\frac{80352496735}{4519258552}t^5h^4+\frac{47763496271}{5649073190}t^6h^4-\frac{7882574121}{3954351233}t^7h^4+\frac{830542815}{4519258552}t^8h^4\end{cases}(38)$$

$$\beta_3''(t)=\begin{cases}-\frac{17675364002}{14413610244285}h^4\frac{99316143581}{14413610244285}th^4-\frac{547765327}{10168331742}t^3h^4+\frac{1608095383}{13557775656}h^4t^4\\-\frac{1555317659}{13557775656}t^5h^4+\frac{323588393}{5649073190}t^6h^4-\frac{113933499}{7908702466}t^7h^4+\frac{3262437}{2259629276}t^8h^4\end{cases}(39)$$

The third derivative give:

$$\alpha_0'''(t)=\begin{cases}-\frac{2403550179}{24628112}+\frac{1378138050}{1539257}t^2-\frac{2856218355}{1539257}t^3+\frac{20746308465}{12314056}t^4-\frac{2449284723}{3078514}t^5\\+\frac{1171827405}{6157028}t^6-\frac{55801305}{3078514}t^7\end{cases}(40)$$

$$\alpha_{\frac{1}{3}}'''(t)=\begin{cases}\frac{1528524158400}{3954351233}-\frac{2482816454352}{564907319}t^2+\frac{5709633757488}{564907319}t^3-\frac{5770196909442}{564907319}t^4\\+\frac{15090236935668}{2824536595}t^5-\frac{789958860444}{564907319}t^6+\frac{81005071572}{564907319}t^7\end{cases}\quad(41)$$

$$\alpha_{\frac{3}{3}}'''(t)=\begin{cases}-\frac{2663117652339}{4519258552}+\frac{4734715377546}{564907319}t^2-\frac{11886655536504}{564907319}t^3+\frac{51662972889801}{2259629276}t^4\\-\frac{35726282304111}{2824536595}t^5+\frac{3898212067953}{1129814638}t^6-\frac{205827276447}{564907319}t^7\end{cases}\quad(42)$$

$$\alpha_1'''(t)=\begin{cases}\frac{208728104616}{564907319}-\frac{3611922485220}{564907319}t^2+\frac{9659692884996}{564907319}t^3-\frac{10965523906512}{564907319}t^4\\+\frac{31221160240158}{2824536595}t^5-\frac{1735440977088}{564907319}t^6+\frac{185478223410}{564907319}t^7\end{cases}\quad(43)$$

$$\alpha_2'''(t)=\begin{cases}-\frac{1086622566093}{4519258552}+\frac{3075893128602}{564907319}t^2-\frac{8848904893668}{564907319}t^3+\frac{41778370413423}{2259629276}t^4\\-\frac{30290218955073}{2824536595}t^5+\frac{3393061454223}{1129814638}t^6-\frac{181772485317}{564907319}t^7\end{cases}\quad(44)$$

$$\alpha_{\frac{7}{3}}'''(t)=\begin{cases}\frac{137880657144}{564907319}-\frac{3166964230716}{564907319}t^2+\frac{9137699369244}{564907319}t^3-\frac{10778541849414}{564907319}t^4\\+\frac{31178421851166}{2824536595}t^5-\frac{1740902064804}{564907319}t^6+\frac{185998327002}{564907319}t^7\end{cases}\quad(45)$$

$$\alpha'''_{\frac{8}{3}}(t)=\begin{Bmatrix}-\frac{4605484383621}{63269619728}+\frac{945317999790}{564907319}t^2-\frac{2723233445271}{564907319}t^3+\frac{25617519509841}{4519258552}t^4\\-\frac{18452198068911}{5649073190}t^5+\frac{2052599907357}{2259629276}t^6-\frac{109284641505}{1129814638}t^7\end{Bmatrix} \quad (46)$$

$$\beta'''_0(t)=\begin{Bmatrix}-\frac{4321725420727}{28827220488570}h^4-\frac{16836604981}{6778887828}t^2h^4+\frac{5243265089}{1694721957}t^3h^4-\frac{28979559035}{13557775656}h^4t^4\\+th^4+\frac{2349102141}{2824536595}t^5h^4-\frac{96290799}{564907319}t^6h^4+\frac{8071083}{564907319}t^7h^4\end{Bmatrix} \quad (47)$$

$$\beta'''_1(t)=\begin{Bmatrix}\frac{2161991632091}{4804536748095}h^4-\frac{3207042597}{1129814638}t^2h^4-\frac{8033333893}{3389443914}t^3h^4+\frac{61103912345}{4519258552}h^4t^4\\-\frac{36296134563}{2824536595}t^5h^4+\frac{5284329579}{1129814638}t^6h^4-\frac{331456212}{564907319}t^7h^4\end{Bmatrix} \quad (48)$$

$$\beta'''_2(t)=\begin{Bmatrix}\frac{11150475106553}{9609073496190}h^4-\frac{60048478941}{2259629276}t^2h^4-\frac{129096637516}{1694721957}t^3h^4+\frac{401762483675}{4519258552}h^4t^4\\+\frac{143290488813}{2824536595}t^5h^4-\frac{7882574121}{564907319}t^6h^4+\frac{830542815}{564907319}t^7h^4\end{Bmatrix} \quad (49)$$

$$\beta'''_3(t)=\begin{Bmatrix}\frac{99316143581}{14413610244285}h^4-\frac{547765327}{3389443914}t^2h^4+\frac{1608095383}{3389443914}t^3h^4-\frac{7776588295}{13557775656}h^4t^4\\+\frac{970765179}{2824536595}t^5h^4-\frac{113933499}{1129814638}t^6h^4+\frac{6524874}{564907319}t^7h^4\end{Bmatrix} \quad (50)$$

Evaluating at non interpolation point gives the main scheme of the method.

$$y_{n+3}=\begin{cases}\frac{1}{1835}\begin{pmatrix}24309y_{n+\frac{8}{3}}-70218y_{n+\frac{7}{3}}+65814y_{n+2}-65814y_{n+1}+70218y_{n+\frac{2}{3}}\\-24309y_{n+\frac{1}{3}}\end{pmatrix}-\\+\frac{h^4}{1337715}(202f_{n+3}-151806f_{n+2}+151806f_{n+1}-202f_n)\end{cases} \quad (51)$$

Putting t = 1 in (7) and evaluate its first, second and third derivatives at points $x = x_n, x_{n+\frac{1}{3}}, x_{n+1}, x_{n+2}, x_{n+\frac{7}{3}}, x_{n+\frac{8}{3}}, x_{n+3}$.

While fourth derivative of (7) is evaluated at points $x = x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+\frac{7}{3}}$ to produce the following discrete schemes represented in matrix form:

$$Y_m = A_i y_i + h^4 b_i f_i \quad (52)$$

Where

$$Y_m = \begin{bmatrix} y_{n+3} \\ hy'_n \\ hy'_{n+\frac{1}{3}} \\ hy'_{n+\frac{2}{3}} \\ hy'_{n+1} \\ hy'_{n+2} \\ hy'_{n+\frac{7}{3}} \\ hy'_{n+\frac{8}{3}} \\ hy'_{n+3} \\ h^2y''_n \\ h^2y''_{n+\frac{1}{3}} \\ h^2y''_{n+\frac{2}{3}} \\ h^2y''_{n+1} \\ h^2y''_{n+2} \\ h^2y''_{n+\frac{7}{3}} \\ h^2y''_{n+\frac{8}{3}} \\ h^2y''_{n+3} \\ h^3y'''_n \\ h^3y'''_{n+\frac{1}{3}} \\ h^3y'''_{n+\frac{2}{3}} \\ h^3y'''_{n+1} \\ h^3y'''_{n+2} \\ h^3y'''_{n+\frac{7}{3}} \\ h^3y'''_{n+\frac{8}{3}} \\ h^3y'''_{n+3} \\ h^4y^{i=1}_{n+\frac{1}{3}} \\ h^4y^{i=1}_{n+\frac{2}{3}} \\ h^4y^{i=1}_{n+\frac{7}{3}} \end{bmatrix}, y_i = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+\frac{7}{3}} \\ y_{n+\frac{8}{3}} \end{bmatrix}, f_i = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

$b_i = h^4$	$-\frac{202}{1337715}\frac{151806}{1337715}-\frac{151806}{1337715}\frac{202}{1337715}$ $-\frac{6180594428}{6177261533265}\frac{19893911528}{2059087177755}\frac{33917903872}{2059087177755}\frac{599647448}{617726153326}$ $\frac{228288085}{1647269742204}-\frac{5039739311}{1647269742204}-\frac{5692761713}{1647269742204}-\frac{3712669}{183029971}$ $-\frac{428058812}{8648166146571}\frac{28816148}{11863053699}\frac{4200113954}{2882722048857}\frac{75448148}{8648166146571}$ $\frac{741383173}{17296332293142}-\frac{27094619089}{5765444097714}-\frac{4817077595}{5765444097714}-\frac{1071279}{1729633229}$ $-\frac{102091240}{8648166146571}\frac{41400934588}{2882722048857}-\frac{25445086246}{2882722048857}-\frac{21503639}{8648166146}$ $\frac{31046567}{2470904613306}-\frac{14377389739}{823634871102}\frac{1241854799}{91514985678}\frac{69699337}{2470904613306}$ $-\frac{8308}{124679817}\frac{77175848}{1122118353}-\frac{69864880}{1122118353}-\frac{57976}{1122118353}$ $-\frac{196603620851}{172963322931420}\frac{44094468680291}{57654440977140}-\frac{45601199511491}{57654440977140}\frac{35287013}{1729633229}$ $\frac{45859834369}{2882722048857}-\frac{501637488494}{4804536748095}-\frac{997131996292}{4804536748095}-\frac{176753640}{1441361024}$ $-\frac{10398301183}{8648166146571}\frac{56275737751}{4804536748095}\frac{343622412449}{14413610244285}\frac{6044583824}{432408307328}$ $-\frac{8308}{124679817}\frac{77175848}{1122118353}-\frac{69864880}{1122118353}-\frac{57976}{1122118353}$ $-\frac{196603620851}{172963322931420}\frac{44094468680291}{57654440977140}$ $-\frac{45601199511491}{57654440977140}\frac{352870136291}{172963322931420}$ $\frac{45859834369}{2882722048857}-\frac{501637488494}{4804536748095}-\frac{997131996292}{4804536748095}$ $-\frac{17675364002}{14413610244285}$ $-\frac{10398301183}{8648166146571}\frac{56275737751}{4804536748095}\frac{343622412449}{14413610244285}\frac{6044583824}{43240830732855}$ $\frac{31046567}{2470904613306}-\frac{14377389739}{823634871102}\frac{1241854799}{91514985678}\frac{69699337}{2470904613306}$
	$\frac{8001478712}{43240830732855}\frac{36228202826}{14413610244285}-\frac{26053231903}{4804536748095}$ $-\frac{1282808978}{43240830732855}$
	$\frac{83189033}{711783221940}-\frac{25529687167}{711783221940}\frac{584791327}{142356644388}\frac{1563773}{237261073980}$

	$\frac{194611274}{2882722048857} - \frac{200220958174}{4804536748095} \frac{47632277083}{4804536748095} \frac{806520758}{14413610244285}$
	$\frac{12280336393}{172963322931420} - \frac{1558107947047}{19218146992380} \frac{4506597869609}{57654440977140} \frac{14594342543}{172963322931420}$
	$-\frac{42143339629}{43240830732855} \frac{12414935791502}{14413610244285} - \frac{793499077720}{960907349619} - \frac{3803582462}{43240830732855}$
	$-\frac{90301376836}{14413610244285} \frac{17196043753342}{4804536748095} - \frac{18694813238128}{4804536748095} \frac{301925184679}{14413610244285}$
	$-\frac{4321725420727}{28827220488570} \frac{2161991632091}{4804536748095} \frac{11150475106553}{9609073496190} \frac{99316143581}{1441361024428}$
	$-\frac{26635149917}{23061776390856} \frac{1283785828337}{7687258796952} \frac{936157904131}{7687258796952} \frac{16276294529}{23061776390856}$
	$\frac{79795543967}{28827220488570} - \frac{764006821441}{4804536748095} - \frac{949362394093}{9609073496190} - \frac{8351190571}{14413610244285}$
	$-\frac{195726018289}{115308881954280} \frac{85924534189}{38436293984760} \frac{4175850553871}{38436293984760} \frac{613962452}{11530888195}$
	$-\frac{15456773}{5765444097714} - \frac{486961849061}{960907349619} \frac{760834943719}{1921814699238} \frac{3365972713}{2882722048857}$
	$\left[-\frac{34230302257}{115308881954280} \frac{29108426393197}{38436293984760} - \frac{19198922245297}{38436293984760} - \frac{218142349043}{115308881954280}\right]$

$$\begin{bmatrix} -\frac{233904530791}{28827220488570} & \frac{26163715041083}{4804536748095} & -\frac{55102359747751}{9609073496190} & \frac{123426550013}{14413610244285} \\ -\frac{2493839125201}{115308881954280} & \frac{381083661125941}{38436293984760} & -\frac{442981494608881}{38436293984760} & \frac{18986211659461}{115308881954280} \\ \frac{4855174760}{45757492839} & -\frac{6928209820}{5084165871} & -\frac{41085711752}{15252497613} & -\frac{719846285}{45757492839} \\ -\frac{1107861230}{45757492839} & -\frac{4224040772}{15252497613} & \frac{3836207557}{5084165871} & \frac{190752452}{45757492839} \\ -\frac{292055020}{45757492839} & \frac{14717142727}{1694721957} & -\frac{125169702644}{15252497613} & -\frac{625053758}{45757492839} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{142913475}{24628112} & -\frac{11701378656}{19771756165} & \frac{753734789793}{22596292760} & -\frac{98423077608}{2824536595} & \frac{520104622239}{22596292760} & -\frac{60387367896}{2824536595} & \frac{1975963772871}{316348098640} \\ \frac{86403267}{24628112} & -\frac{557389489554}{19771756165} & \frac{1157426146287}{22596292760} & -\frac{73780538832}{2824536595} & -\frac{301301267679}{22596292760} & \frac{54272893446}{2824536595} & -\frac{1992510149031}{316348098640} \\ -\frac{86403267}{24628112} & \frac{1043446829472}{19771756165} & -\frac{3467715448671}{22596292760} & \frac{393071696856}{2824536595} & -\frac{2253027996513}{22596292760} & \frac{234513269352}{2824536595} & -\frac{5784407289657}{316348098640} \\ \frac{142913475}{24628112} & -\frac{1643409338706}{19771756165} & \frac{5500648136943}{22596292760} & -\frac{652869793008}{2824536595} & \frac{5490238342689}{22596292760} & -\frac{721410497946}{2824536595} & \frac{24505807704921}{316348098640} \\ \frac{1208255517}{24628112} & -\frac{12698581610304}{19771756165} & \frac{41831888313537}{22596292760} & -\frac{4892309627832}{2824536595} & \frac{37632319939071}{22596292760} & -\frac{4730440907304}{2824536595} & \frac{152906408047719}{316348098640} \\ \frac{2403550179}{24628112} & -\frac{24122992949442}{19771756165} & \frac{78871016948751}{22596292760} & -\frac{9207564113736}{2824536595} & \frac{70744501555713}{22596292760} & -\frac{8883831871602}{2824536595} & \frac{286713376437177}{316348098640} \\ \frac{282247200}{1539257} & -\frac{439271812800}{564907319} & \frac{712786314240}{564907319} & -\frac{468195729120}{564907319} & \frac{314498419200}{564907319} & -\frac{318511167840}{564907319} & \frac{95109253920}{564907319} \\ \frac{107556120}{1539257} & -\frac{129925410048}{564907319} & \frac{128129099616}{564907319} & -\frac{22568601888}{564907319} & -\frac{73063553472}{564907319} & \frac{83888620704}{564907319} & -\frac{25933250952}{564907319} \\ \frac{107556120}{1539257} & -\frac{548849595168}{564907319} & \frac{1594363747536}{564907319} & -\frac{1488803249808}{564907319} & \frac{1393171094448}{564907319} & -\frac{1382346027216}{564907319} & \frac{392990934168}{564907319} \end{bmatrix}$$

Adopting matrix inversion method to solve (53), $y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+2}, y_{n+\frac{7}{3}}, y_{n+\frac{8}{3}}, y'_{n+\frac{1}{3}}, y'_{n+\frac{2}{3}}, y'_{n+1}, y'_{n+2}, y'_{n+\frac{7}{3}}, y'_{n+\frac{8}{3}}, y'_{n+3}, y''_{n+\frac{1}{3}}, y''_{n+\frac{2}{3}}, y''_{n+1}$, $y''_{n+2}, y''_{n+\frac{7}{3}}, y''_{n+\frac{8}{3}}, y''_{n+3}, y'''_{n+\frac{1}{3}}, y'''_{n+\frac{2}{3}}, y'''_{n+1}, y'''_{n+2}, y'''_{n+\frac{7}{3}}, y'''_{n+\frac{8}{3}}, y'''_{n+3}$ are determine and expressed as given below

$$y_{n+\frac{1}{3}} = \left\{ \begin{array}{c} \frac{1}{111106598400}\Big(-76015f_{n+3} + 1064502f_{n+\frac{7}{3}} - 1944348f_{n+2} + 10191930f_{n+1} \\ -24838812f_{n+\frac{2}{3}} + 35431263f_{n+\frac{1}{3}} + 37325080f_n\Big)h^4 + \frac{1}{162}y'''_n h^3 \\ + \frac{1}{18}y''_n h^2 + \frac{1}{3}y'_n h + y_n \end{array} \right\}$$

$$y_{n+\frac{2}{3}} = \begin{cases} \frac{1}{434010150}\Big(-4535f_{n+3} + 63666f_{n+\frac{7}{3}} - 116508f_{n+2} + 618870f_{n+1} \\ -1520856f_{n+\frac{2}{3}} + 2908143f_{n+\frac{1}{3}} + 1623320f_n\Big)h^4 \\ +\frac{4}{81}y'''_{n}h^3 + \frac{2}{9}y''_{n}h^2 + \frac{2}{3}y'_{n}h + y_n \end{cases} \quad (53)$$

$$y_{n+1} = \begin{cases} \frac{1}{5644800}\Big(-235f_{n+3} + 3294f_{n+\frac{7}{3}} - 6020f_{n+2} + 31570f_{n+1} - 65124f_{n+\frac{2}{3}} \\ +191835f_{n+\frac{1}{3}} + 79880f_n\Big)h^4 + \frac{1}{6}y'''_{n}h^3 + \frac{1}{2}y''_{n}h^2 + y'_{n}h + y_n \end{cases} \quad (54)$$

$$y_{n+2} = \begin{cases} = \frac{1}{22050}\Big(-5f_{n+3} + 54f_{n+\frac{7}{3}} - 56f_{n+2} + 3010f_{n+1} - 324f_{n+\frac{2}{3}} \\ +9261f_{n+\frac{1}{3}} + 2760f_n\Big)h^4 + \frac{4}{3}y'''_{n}h^3 + 2y''_{n}h^2 + 2y'_{n}h + y_n \end{cases} \quad (55)$$

$$y_{n+\frac{7}{3}} = \begin{cases} \frac{1}{2267481600}\Big(84035f_{n+3} - 8946126f_{n+\frac{7}{3}} + 43765428f_{n+2} \\ +737659230f_{n+1} - 70791084f_{n+\frac{2}{3}} + 1648615437f_{n+\frac{1}{3}} + 450139480f_n\Big)h^4 \\ +\frac{343}{162}y'''_{n}h^3 + \frac{49}{18}y''_{n}h^2 + \frac{7}{3}y'_{n}h + y_n \end{cases}$$

$$y_{n+\frac{8}{3}} = \begin{cases} \frac{1}{217005075}\Big(181760f_{n+3} - 4230144f_{n+\frac{7}{3}} + 18600960f_{n+2} \\ +143969280f_{n+1} - 18994176f_{n+\frac{2}{3}} + 254016000f_{n+\frac{1}{3}} + 63685120f_n\Big)h^4 \\ +\frac{256}{81}y'''_{n}h^3 + \frac{32}{9}y''_{n}h^2 + \frac{8}{3}y'_{n}h + y_n \end{cases}$$

$$y_{n+3} = \begin{cases} \frac{1}{627200}\Big(1755f_{n+3} - 21870f_{n+\frac{7}{3}} + 138348f_{n+2} + 754110f_{n+1} \\ -131220f_{n+\frac{2}{3}} + 1117557f_{n+\frac{1}{3}} + +258120f_n\Big)h^4 + \frac{9}{2}y'''_{n}h^3 \\ +\frac{9}{2}y''_{n}h^2 + 3y'_{n}h + y_n \end{cases}$$

$$y'_{n+\frac{1}{3}} = \begin{cases} \frac{1}{12345177600}\Big(-107665f_{n+3} + 1509786f_{n+\frac{7}{3}} - 2760576f_{n+2} \\ +14588070f_{n+1} - 35900496f_{n+\frac{2}{3}} + 54051921f_{n+\frac{1}{3}} + 44823760f_n\Big)h^3 \\ +\frac{1}{18}y'''_{n}h^2 + \frac{1}{3}y''_{n}h + y'_{n} \end{cases}$$

$$y'_{n+\frac{2}{3}} = \begin{cases} \frac{1}{48223350}\Big(-2705f_{n+3} + 37962f_{n+\frac{7}{3}} - 69447f_{n+2} + 366870f_{n+1} \\ -877797f_{n+\frac{2}{3}} + 2020977f_{n+\frac{1}{3}} + 905540f_n\Big)h^3 + \frac{2}{9}y'''_{n}h^2 + \frac{2}{3}y''_{n}h + y'_{n} \end{cases}$$

$$y'_{n+1} = \left\{ \begin{array}{c} \frac{1}{5644800}\Big(-775f_{n+3} + 10854f_{n+\frac{7}{3}} - 19824f_{n+2} + 105210f_{n+1} \\ -147744f_{n+\frac{2}{3}} + 735399f_{n+\frac{1}{3}} + 257680f_n\Big)h^3 + 1/2y'''_n h^2 + y''_n h + y'_n \end{array} \right\}$$

$$y'_{n+2} = \left\{ \begin{array}{c} \frac{1}{22050}\Big(5f_{n+3} - 162f_{n+\frac{7}{3}} + 567f_{n+2} + 8610f_{n+1} - 243f_{n+\frac{2}{3}} \\ +16443f_{n+\frac{1}{3}} + 4180f_n\Big)h^3 + 2y'''_n h^2 + 2y''_n h + y'_n \end{array} \right\} \quad (62)$$

$$y'_{n+\frac{8}{3}} = \left\{ \begin{array}{c} \frac{1}{24111675}\Big(84800f_{n+3} - 1351296f_{n+\frac{7}{3}} + 7061376f_{n+2} \\ +31032960f_{n+1} - 6044544f_{n+\frac{2}{3}} + 37727424f_{n+\frac{1}{3}} + 7694080f_n\Big)h^3 \\ +\frac{32}{9}y'''_n h^2 + \frac{8}{3}y''_n h + y'_n \end{array} \right\} \quad (64)$$

$$y'_{n+3} = \left\{ \begin{array}{c} \frac{1}{627200}\Big(6165f_{n+3} - 13122f_{n+\frac{7}{3}} + 326592f_{n+2} + 1241730f_{n+1} \\ -314928f_{n+\frac{2}{3}} + 1331883f_{n+\frac{1}{3}} + 244080f_n\Big)h^3 + \frac{9}{2}y'''_n h^2 + 3y''_n h + y'_n \end{array} \right\} \quad (65)$$

$$y''_{n+\frac{1}{3}} = \left\{ \begin{array}{c} \frac{1}{45722880}\Big(-3673f_{n+3} + 51606f_{n+\frac{7}{3}} - 94500f_{n+2} + 505218f_{n+1} \\ -1261332f_{n+\frac{2}{3}} + 2092545f_{n+\frac{1}{3}} + 1250296f_n\Big)h^2 + \frac{1}{3}y'''_n h + y''_n \end{array} \right\} \quad (66)$$

$$y''_{n+\frac{1}{3}} = \left\{ \begin{array}{c} \frac{1}{45722880}\Big(-3673f_{n+3} + 51606f_{n+\frac{7}{3}} - 94500f_{n+2} + 505218f_{n+1} \\ -1261332f_{n+\frac{2}{3}} + 2092545f_{n+\frac{1}{3}} + 1250296f_n\Big)h^2 + \frac{1}{3}y'''_n h + y''_n \end{array} \right\} \quad (67)$$

$$y''_{n+\frac{2}{3}} = \left\{ \begin{array}{c} \frac{1}{2143260}\Big(-413f_{n+3} + 5778f_{n+\frac{7}{3}} - 10542f_{n+2} + 53886f_{n+1} \\ -109998f_{n+\frac{2}{3}} + 402381f_{n+\frac{1}{3}} + 135188f_n\Big)h^2 + \frac{2}{3}y'''_n h + y''_n \end{array} \right\} \quad (68)$$

$$y''_{n+1}=\begin{Bmatrix}\frac{1}{564480}\Big(-173f_{n+3}+2430f_{n+\frac{7}{3}}-4452f_{n+2}+27930f_{n+1}\\ +8748f_{n+\frac{2}{3}}+192213f_{n+\frac{1}{3}}+55544f_n\Big)h^2+y'''_nh+y''_n\end{Bmatrix} \quad (67)$$

$$y''_{n+2}=\begin{Bmatrix}\frac{1}{8820}\Big(23f_{n+3}-486f_{n+\frac{7}{3}}+1470f_{n+2}+8022f_{n+1}\\ -1458f_{n+\frac{2}{3}}+8505f_{n+\frac{1}{3}}+1564f_n\Big)h^2+2y'''_nh+y''_n\end{Bmatrix} \quad (69)$$

$$y''_{n+\frac{7}{3}}=\begin{Bmatrix}\frac{1}{933120}\Big(4459f_{n+3}-85554f_{n+\frac{7}{3}}+374556f_{n+2}+1253322f_{n+1}\\ -333396f_{n+\frac{2}{3}}+1145277f_{n+\frac{1}{3}}+181496f_n+\Big)h^2+\frac{7}{3}y'''_nh+y''_n\end{Bmatrix} \quad (69)$$

$$y''_{n+\frac{8}{3}}=\begin{Bmatrix}\frac{1}{535815}\Big(4576f_{n+3}-8640f_{n+\frac{7}{3}}+337344f_{n+2}+960960f_{n+1}\\ -305856f_{n+\frac{2}{3}}+804384f_{n+\frac{1}{3}}+112352f_n\Big)h^2+\frac{8}{3}y'''_nh+y''_n\end{Bmatrix} \quad (70)$$

$$y''_{n+3}=\begin{Bmatrix}\frac{1}{62720}\Big(2415f_{n+3}+16038f_{n+\frac{7}{3}}+44604f_{n+2}+150066f_{n+1}\\ -61236f_{n+\frac{2}{3}}+117369f_{n+\frac{1}{3}}+12984f_n\Big)h^2+3y'''_nh+y''_n\end{Bmatrix} \quad (71)$$

$$y''''_{n+\frac{1}{3}}=\begin{Bmatrix}\frac{1}{22861440}\Big(-9691f_{n+3}+136566f_{n+\frac{7}{3}}-250656f_{n+2}+1364874f_{n+1}\\ -3488832f_{n+\frac{2}{3}}+7328475f_{n+\frac{1}{3}}+2539744f_n\Big)h+y'''_n\end{Bmatrix} (71)$$

$$y'''_{n+\frac{2}{3}}=\begin{Bmatrix}\frac{1}{1428840}\Big(-367f_{n+3}+5022f_{n+\frac{7}{3}}-8988f_{n+2}+34818f_{n+1}\\ +96876f_{n+\frac{2}{3}}+676431f_{n+\frac{1}{3}}+148768f_n\Big)h+y'''_n\end{Bmatrix} \quad (73)$$

$$y'''_{n+1}=\begin{Bmatrix}\frac{1}{94080}\Big(-41f_{n+3}+594f_{n+\frac{7}{3}}-1120f_{n+2}+16814f_{n+1}\\ +25920f_{n+\frac{2}{3}}+41769f_{n+\frac{1}{3}}+10144f_n\Big)h+y'''_n\end{Bmatrix} \quad (75)$$

$$y'''_{n+2}=\begin{Bmatrix}\frac{1}{5880}\Big(43f_{n+3}-918f_{n+\frac{7}{3}}+3388f_{n+2}7798f_{n+1}\\ -3564f_{n+\frac{2}{3}}+4725f_{n+\frac{1}{3}}+288f_n\Big)h+y'''_n\end{Bmatrix} \quad (76)$$

$$y'''_{n+\frac{7}{3}}=\begin{Bmatrix}\frac{1}{466560}\Big(2597f_{n+3}-3402f_{n+\frac{7}{3}}+362208f_{n+2}+594762f_{n+1}\\ -254016f_{n+\frac{2}{3}}+361179f_{n+\frac{1}{3}}+25312f_n+\Big)h+y'''_n\end{Bmatrix} \quad (76)$$

$$y'''_{n+\frac{8}{3}}=\begin{Bmatrix}\frac{1}{178605}\Big(+4804f_{n+3}+92664f_{n+\frac{7}{3}}+89880f_{n+2}+270984f_{n+1}\\ -152280f_{n+\frac{2}{3}}+165564f_{n+\frac{1}{3}}+4664f_n\Big)h+y'''_n\end{Bmatrix} \quad (77)$$

3 Properties of Method

In this section, analysis of the derived schemes such as order and error constant, consistency, zero-stability, convergence and the stability domain was carried out. Suppose the linear Operator defined on the method (7) be defined as,

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h^4 \beta_j y^{iv}(x \tag{79}$$

where $y(x)$ is any continuously differentiable test function on the interval $[a,b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $y'''(x_n + jh), j = 0,1,\dots,k$ in (52). Taylor series about x_n and collecting like terms in h and y gives; $[y(x);h] = I_0 y(x) + I_1 h y'(x) + \quad \dots I_p h^p y^p(x) + I_{p+1} h^{p+1} y^{p+1}(x) + \quad I_{p+2} h^{p+2} y^{p+2}(x) + \cdots$

Definition 1:According to Ref. [20] The term I_{p+4} is called an error constant, meaning that the local truncation error is given as $T_{n+k} = I_{p+4} h^{(p+4)} y^{p+4}(x) + 0(h^{p+5})$

Definition 2. The difference operator L associated with the hybrid block method with the step number three (52) are said to be of order p if $I_0 = I_1 = I_2 = \cdots = I_{p+3} = 0, I_{p+4} \neq 0$ see Ref.[23]

Definition 3. According to Ref. [21,28]

LMM is a computational method for determining the sequence y_n which takes the form of a linear relationship between y_{n+j} and $f_{n+j}, j = 0(1)k$. The general form of a linear k-step method for mth order general odes may be written as

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j} \tag{80}$$

α_j, β_j are the coefficients of the method, $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y_{n+j}^{i}, y_{n+j}^{ii}, \dots, y_{n+j}^{m-1}\right)$, $j_{+\overline{jh}}\rho(1)k, h$ is the step length, m is the order of ode to be solved: $\alpha_k \neq 0. \alpha_0$ and β_0 are not both zero. According to Ref. [26,27]

Definition 4. A multi-step method is said to be P-satble, if its interval of periodicity is $(0,\infty)$ seeRef. [24]

3.1 Order and Error constant of the Method

Apply the linear operator L in (80) to determine the order and error constant of the derived method. Expanding the method (52), the additional methods and its derivatives in Taylor series and combining the coefficients of the same terms with h^n yields $I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = 0, I_{11} = \frac{11759}{39007769400}$ Similarly, the additional methods and their derivatives were also analyzed and summarized below:

Table 1: The order and error constants

Equation number	Order	Error Constant
(54)	7	$\frac{1462493}{148493968761600}$
(55)	7	$\frac{86518}{580054565475}$
(56)	7	$\frac{643}{1077753600}$
(57)	7	$\frac{2}{467775}$
(58)	7	$\frac{11781707}{3030489158400}$
(59)	7	$\frac{68608}{580054565475}$
(60)	7	$\frac{97}{10348800}$
(61)	7	$\frac{8927}{71425670400}$
(62)	7	$\frac{4694}{5859137025}$
(63)	7	$\frac{451}{228614400}$
(64)	7	$\frac{2}{893025}$
(65)	7	$\frac{55223}{10203667200}$
(66)	7	$\frac{2176}{119574225}$
(67)	7	$\frac{13}{313600}$
(68)	7	$\frac{285989}{249989846400}$
(69)	7	$\frac{10837}{3906091350}$
(70)	7	$\frac{167}{38102400}$
(71)	7	$\frac{1}{66150}$

(72)	7	$-\frac{154693}{5101833600}$
(73)	7	$-\frac{95296}{1953045675}$
(74)	7	$-\frac{137}{1411200}$
(75)	7	$-\frac{949}{158723712}$
(76)	7	$-\frac{97}{24800580}$
(77)	7	$-\frac{19}{3265920}$
(78)	7	$-\frac{1}{20412}$
(79)	7	$-\frac{4753}{113374080}$
(80)	7	$-\frac{536}{6200145}$

Zero-stability of the Method

Suppose the first characteristics polynomial of equation (52) is, $\rho(r)=r^3+\frac{24309}{1835}r^{\frac{8}{3}}-\frac{70218}{1835}r^{\frac{7}{3}}+\frac{70218}{1835}r^{\frac{2}{3}}-\frac{24309}{1835}\sqrt[3]{r}-\frac{65814}{1835}r+\frac{65814}{1835}r^2-1$

Solving $\rho(r), r=0,1,1$ which satisfies $|R_J|\geq 1, j=1,\ldots,k$. The roots are inside the unit circle and the multiplicity is simple. Therefore, the method is zero stable.

Consistency of the Method

According to Ref. [23], the major condition is sufficient for method to be consistence is to have an order p equals or greater than one. Consequently, Our method is of order $p=7$. So it's consistent.

Convergence of the Method

A numerical method is said to be convergence, If it is consistence and zerostable. Thus, the methods are convergence since it satisfied the conditions in Section 3.2 and Section 3.3.

Stability Domain of the Method

The region of absolute stability of the method is examined via the procedure discussed in Lambert (1973) and Ibijola et al. (2011). The stability matrix can be expressed as

$$M(z)=zB(I-zA)^{-1}U+V \tag{81}$$

together with the Stability function

$$p(n,z)=\det(-M(z)+nI) \tag{82}$$

for the Stability properties, the method (52) - (79) was formulated as a general linear method of the form,

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ --- & --- \\ B & --- \\ & V \end{bmatrix} \begin{bmatrix} h^4 f(u) \\ --- \\ Y_{i-1} \end{bmatrix} \qquad (83)$$

where n represents the roots of the first characteristics polynomial, and

$$Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_n \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+3} \end{bmatrix}$$

$$A = \begin{bmatrix} \{0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{933127}{2777664960} & -\frac{8927}{27993600} & -\frac{76663}{342921600} & \frac{48533}{529079040} & -\frac{23147}{1322697600} & \frac{6571}{685843200} & 0 & \frac{15203}{22221319680} \\ \frac{162332}{43401015} & \frac{5129}{765450} & -\frac{9388}{2679075} & \frac{421}{295245} & -\frac{2774}{10333575} & \frac{131}{893025} & 0 & -\frac{907}{86802030} \\ \frac{1997}{141120} & \frac{87}{2560} & -\frac{1809}{156800} & \frac{451}{80640} & -\frac{43}{40320} & \frac{183}{313600} & 0 & -\frac{47}{1128960} \\ \frac{92}{735} & -\frac{21}{50} & -\frac{18}{1225} & \frac{43}{315} & -\frac{4}{1575} & \frac{3}{1225} & 0 & -\frac{1}{4410} \\ \frac{11253487}{56687040} & \frac{20353277}{27993600} & -\frac{218491}{6998400} & \frac{24588641}{75582720} & \frac{3647119}{188956800} & -\frac{55223}{13996800} & 0 & \frac{16807}{453496320} \\ \frac{12737024}{43401015} & \frac{2560}{2187} & -\frac{234496}{2679075} & \frac{1371136}{2066715} & \frac{177152}{2066715} & -\frac{17408}{893025} & 0 & \frac{36352}{43401015} \\ \frac{6453}{15680} & \frac{159651}{89600} & -\frac{6561}{31360} & \frac{1539}{1280} & \frac{4941}{22400} & -\frac{2187}{62720} & 0 & \frac{351}{125440} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{12737024}{43401015} & \frac{2560}{2187} & -\frac{234496}{2679075} & \frac{1371136}{2066715} & \frac{177152}{2066715} & -\frac{17408}{893025} & 0 & \frac{36352}{43401015} \\ \frac{6453}{15680} & \frac{159651}{89600} & -\frac{6561}{31360} & \frac{1539}{1280} & \frac{4941}{22400} & -\frac{2187}{62720} & 0 & \frac{351}{125440} \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+\frac{7}{3}} \\ y_{n+\frac{8}{3}} \\ y_{n+3} \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+\frac{7}{3}} \\ f_{n+\frac{8}{3}} \\ f_{n+3} \end{bmatrix}$$

Now, putting the values of the variables A, B, U, V, M and I in equations (3.125) and equation (3.126), to obtained the Stability function.

$$T = \frac{1}{2} \frac{\begin{matrix} 518616\,\eta\,z^6 + 5307472720\,\eta\,z^5 - 1481037642\,z^6 + 9369164831730\,\eta\,z^4 \\ -16110313577785\,z^5 + 2318806283194800\,\eta\,z^3 - 12938873970657030\,z^4 \\ -57670342843680000\,\eta\,z^2 - 6668204001859486800\,z^3 \\ +6581132700387840000\,\eta\,z - 859139232473138400000\,z^2 \\ +514361508667453440000\,\eta - 17366282050226941440000\,z \\ -514361508667453440000) \end{matrix}}{\begin{matrix} 259308\,z^5 + 2653736360\,z^5 + 4684582415865\,z^4 + 1159403141597400\,z^3 \\ -28835171421840000\,z^2 + 3290566350193920000\,z + 257180754333726720000 \end{matrix}} \qquad (84)$$

$$Z = \frac{135}{2}\frac{\begin{array}{l}\mathrm{n}(1831479645364516\, z^{10} - 53079842420210812668\, z^{9} \\ -30442766327486303787805 5\, z^{8} - 6692110485990012973285200\, z^{7} \\ +181244051244164248480803096000\, z^{6} \\ +6477818247266184047874939840000 0\, z^{5} \\ +813040976598860870384944008960000 0\, z^{4} \\ -10120302477084669596177099120640000 00\, z^{3} \\ -40561433966959870327557943246848000000 0\, z^{2} \\ -3273613445739495685864070861291520000000 00\, z \\ -33070970199832347535233641820979200000000 0)\end{array}}{\begin{array}{l}(259308\, z^{6} + 2653736360\, z^{5} \\ +4684582415865\, z^{4} + 1159403141597400\, z^{3} \\ -2883517142184000 0\, z^{2} + 329056635019392000 0\, z + 2571807543337267200000)^{2}\end{array}} \quad (85)$$

The stability polynomial and its first derivatives are then plotted in MATLAB (R2012a) environment. It should be noted that M is 8 by 8 identity matrix. The region of absolute stability (RAS) of the method is displayed in the Figure 1 below;

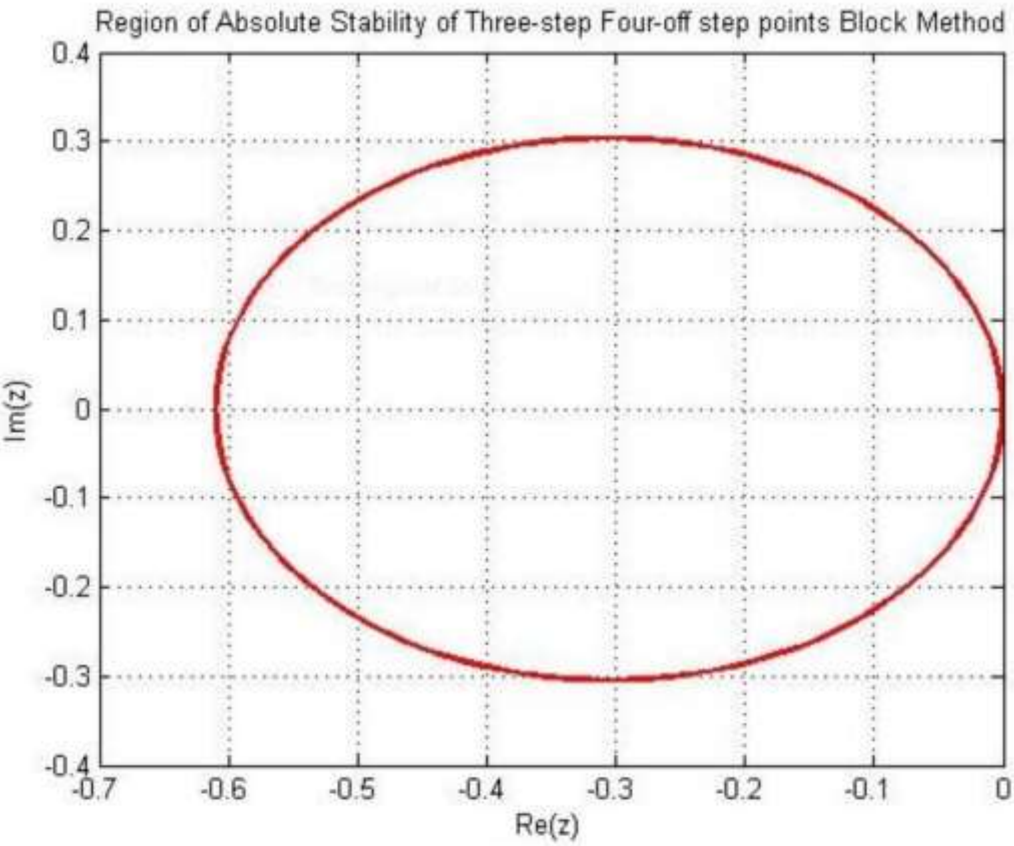

Figure 1: k = 3

The inside of the curve represents the unstable region, while the outer segment of the curve represents the stable region according to Ref.[27]. The new method generated is absolutely-stable.

4 Numerical Examples and Results

Test problems

Problem 1

Linear fourth order problem:

$$y^{iv} = -y''$$
$$y(0) = 0, y'(0) = \frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y'''(0) = \frac{1.2}{144 - 100\pi}, h = \frac{0.1}{32}$$

Exact solution:

Source: Abolarin et al. (2020)

The tables displayed below are the numerical results when the newly proposed block method with step-length 3 with four hybrid points were applied to fourth order differential equations above. The comparison of the generated numerical results is made with the existing methods in terms of error.

Table 3: Numerical Results for Problem 1 using $h = \frac{0.1}{32}$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.00312	0.0000403745930229	0.00001346779354162361	$2.032879e-20$	0.0328
0.00625	0.0000806915800710	0.00008069158007107028	$4.201283e-19$	0.0348
0.00937	0.0001209507467709	0.00012095074677095981	$6.776264e-19$	0.0413
0.01250	0.00016115187931405	0.00013436082766726340	$3.604972e-18$	0.0419
0.01562	0.0002012947644584	0.00020129476445852158	$2.434034e-17$	0.0420
0.01875	0.00024137918953123	0.00024137918953130608	$7.497258e-17$	0.0422
0.02187	0.00028140494243011042	0.00025473091721102804	$1.744481e-16$	0.0425
0.025000	0.00032137181162595881	0.00032137181162630727	$3.484626e-16$	0.0425
0.028125	0.00036127958616463267	0.00036127958616525869	$6.260183e-16$	0.0427
0.031250	0.00040112805566908761	0.00037457232518573017	$1.036389e-15$	0.0430

Here problem 1 is solved using the newly proposed method of order $P = 7$, The results of the problem 1 which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 3. It was also compared with exiting methods namely; Abolarin et al. (2020) who proposed hybrid block method containing higher derivative terms with order $p = 7$. The comparison is shown in Table 4.

Table 4: Comparison of the newly proposed method with Abolarin et al. (2020) for Problem 1.

x-value	Error in New Results, P = 7	Error in Abolarin et al. (2020) , P = 7
0.003125	2.032879e − 20	2.11320e − 18
0.006250	4.201283e − 19	1.05766e − 17
0.009375	6.776264e − 19	4.43030e − 17
0.001250	3.604972e − 18	6.74370e − 17
0.015625	2.434034e − 17	1.15415e − 16
0.018750	7.497258e − 17	1.52114e − 16
0.021875	1.744481e − 16	2.13223e − 16
0.025000	3.484626e − 16	2.62354e − 16
0.028125	6.260183e − 16	3.35455e − 16
0.031250	1.036389e − 15	3.95875e − 16

Problem 2

Homogeneous linear fourth order problem

$y^{iv} = (x^4 + 14x^3 + 49x^2 + 32x + 2)\exp(x)$

$y(0) = 0, y'(0) = 0, y''(0) = 2, y'''(0) = -6, h = 0.01$

Exact solution: $y(x) = x^2(x-1)^2\exp(x)$

Source: Ahamad and Charan (2019), and Jena et al. (2018)

Table 5: Numerical Results for Problem 2 using $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.100000	0.00895188443641274640	0.00895188443641274470	1.734723e − 18	0.0026
0.20000	0.03126791060890035000	0.03126791060890034300	6.938894e − 18	0.0042
0.300000	0.05952877341410175300	0.05952877341410173200	2.081668e − 17	0.0059
0.400000	0.08592910258413720400	0.08592910258413716300	4.163336e − 17	0.0079
0.500000	0.10304507941875804000	0.10304507941875800000	4.163336e − 17	0.0100
0.600000	0.10556148255993897000	0.10556148255993898000	1.387779e − 17	0.0115
0.700000	0.08880649439944791900	0.0888064943994480160	9.714451e − 17	0.0136
0.800000	0.05697384776940699800	0.05697384776940724700	2.498002e − 16	0.0152
0.900000	0.01992278520037110000	0.01992278520037150600	4.059253e − 16	0.0170
1.000000	0.00000000000000000000	0.00000000000000000490	4.908725e − 16	0.0191

Problem 2 is solved using the newly proposed method of order P = 7, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 5. It was also compared with exiting methods namely; Ahamad and Charan (2019) who developed an improved Runge-kutta method, the problem was first converted to system of first order ODEs before solving it. Likewise, Jena et al. (2018) proposed a block algorithm containing nine intermediate steps with order of accuracy of $p = 11$. The comparison is shown in Table 6.

Table 6: Comparison of the newly proposed method with Ahamad and Charan (2019) and Jena et al. (2018) for Problem 2

x-value	**Error in New Results, P = 7**	**Error in Ahamad and Charan (2019), Runge-kutta**	**Error in Jena et al. (2018), P = 11**
0.1	$1.734723e-18$	$6.23665e-08$	$1.5370e-14$
0.2	$6.938894e-18$	$4.71836e-05$	$8.2021e-14$
0.3	$2.081668e-17$	$4.40097e-05$	$3.6666e-13$
0.4	$4.163336e-17$	$7.68036e-05$	$6.3424e-13$
0.5	$4.163336e-17$	$2.17730e-04$	$6.7024e-13$
0.6	$1.387779e-17$	$2.537301e-03$	$5.2608e-13$
0.7	$9.714451e-17$	$7.777177e-03$	$3.3906e-13$
0.8	$2.498002e-16$	$8.830157e-03$	$1.9011e-14$
0.9	$4.059253e-16$	$1.52112e-03$	$9.6152e-14$
1.0	$4.908725e-16$	$0.00000e-00$	$4.4983e-14$

Problem 3

Consider the fourth order below

$y^{iv} = y^2 - yy'' - 4x^2 + \exp(x)(1 - 4x + x^2)$

$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, h = \dfrac{1}{320}$

Exact solution: $y(x) = x^2 + \exp(x)$

Source: Kuboye et al. (2020)

Table 7: : Numerical Results for Problem 3 using $h = \frac{0.1}{32}$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.103125	1.11926474478759190000	1.1192647447886266	1.034728e − 12	0.0599
0.206250	1.27159949319804830000	1.2715994932395991	4.155076e − 11	0.0682
0.306250	1.45211090706501310000	1.4521109073957401	3.307270e − 10	0.0755
0.406250	1.66621686250012350000	1.6614042976157002	1.456810e − 09	0.0835
0.506250	1.91534710992091780000	1.9153471145659231	4.645005e − 09	0.0892
0.603125	2.19158159360620710000	2.1852689263176006	1.174866e − 08	0.0949
0.703125	2.51444029333370000000	2.5144403199643164	2.663062e − 08	0.1029
0.803125	2.87751638774661170000	2.8775164421400721	5.439346e − 08	0.1087
0.903125	3.28293615880510580000	3.2740427190025776	1.025960e − 07	0.1325
1.003125	3.73304951149518250000	3.7330496932018051	1.817066e − 07	0.1385

Problem 3 is a nonlinear fourth-order problem which is solved using the newly proposed method of order P = 7, the numerical results which contains the y -exact results, y-computed results, absolute error and the time of execution is presented in Table 7. It was also compared with exiting methods namely; Kuboye et al. (2020) proposed a block hybrid algorithm with order of accuracy of p = 7. The comparison is shown in Table 8.

Table 8: Comparison of the newly proposed method with Kuboye et al. (2020) for Problem 3.

x-value	Error in New Results, P = 7	Error in Kuboye et al (2020), P=7
0.003125	1.034728e − 12	1.8149238e − 10
0.006250	4.155076e − 11	1.154325e − 08
0.009375	3.307270e − 10	1.2194148e − 07
0.001250	1.456810e − 09	6.5296082e − 07
0.015625	4.645005e − 09	2.3972196e − 06
0.018750	1.174866e − 08	6.7092614e − 06
0.021875	2.663062e − 08	1.6438756e − 05
0.025000	5.439346e − 08	3.5549856e − 05
0.028125	1.025960e − 07	6.9845227e − 05
0.031250	1.817066e − 07	1.2716790e − 04

Problem 4

Special fourth order problem
$y^{iv} = x$
$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0, h = 0.1$
Exact solution: $y(x) = \frac{x^5}{120} + x$
Source: Kuboye et al. (2020) and Duromola (2016)

Table 9: Numerical Results for Problem 4 using $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.100000	0.10000008333333332000	0.09333339235379	0.000000e + 000	0.0793
0.200000	0.20000266666666666000	0.2000026666666	2.775558e − 017	0.0814
0.300000	0.30002025000000004000	0.3000202499999	1.110223e − 016	0.0830
0.400000	0.40008533333333346000	0.39341178867478	2.220446e − 016	0.0850
0.500000	0.50026041666666687000	0.50026041666666	3.330669e − 016	0.0877
0.600000	0.60064800000000029000	0.60064799999999	5.551115e − 016	0.0924
0.700000	0.70140058333333377000	0.69466848055133	7.771561e − 016	0.0942
0.800000	0.80273066666666715000	0.80273066666666	8.881784e − 016	0.0959
0.900000	0.90492075000000061000	0.90492074999999	8.881784e − 016	0.0975
1.000000	1.00833333333333400000	1.00139256798342	8.881784e − 016	0.0995

Problem 4 is a special fourth-order problem which has been solved by scholars such as Kuboye et al. (2020) and Duromola (2016) in the past, the problem was resolved using the newly proposed method of order $\mathrm{P} = 7$, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 9. It was also compared with exiting methods namely; Kuboye et al. (2020) proposed a block hybrid algorithm and Duromola (2016) who proposed a one-step with multiple hybrid points. The two methods compared with both have order of accuracy of $p = 7$. The comparison is shown in Table 10.

Table 10: Comparison of the newly proposed method with Kuboye et al. (2020) and Duromola (2016) for Problem 4.

x-value	Error in New Results, P = 7	Error in Kuboye et al. (2020), p=7	Error in Duromola (2016), P = 7
0.1	$0.000000e+000$	0.00	$1.658e-13$
0.2	$2.775558e-017$	0.00	$3.316e-12$
0.3	$1.110223e-016$	0.00	$7.183e-12$
0.4	$2.220446e-016$	$5.5511151e-17$	$6.649e-11$
0.5	$3.330669e-016$	$1.1102230e-16$	$9.906e-11$
0.6	$5.551115e-016$	$1.1102230e-16$	$3.217e-11$
0.7	$7.771561e-016$	$2.2204460e-16$	$2.432e-10$
0.8	$8.881784e-016$	0.00	$3.202e-10$
0.9	$8.881784e-016$	$1.1102230e-16$	$2.540e-10$
1.0	$8.881784e-016$	$2.2204460e-16$	$2.024e-10$

Problem 5

Non-linear fourth order problem

$y^{iv} = y^2 + \cos^2(x) + \sin(x) - 1y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, h = 0.01$

Exact solution: $y(x) = \sin(x)$

Source: Duromola et. al (2024)

Table 11: Numerical Results for Problem 5 using $h = 0.02$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.020000	0.01999866669333308000	0.01999866669333	$0.000000e+000$	0.0004
0.040000	0.03998933418663416100	0.03332716083676	$0.000000e+000$	0.0009
0.060000	0.05996400647944459500	0.05996400647944	$0.000000e+000$	0.0014
0.080000	0.07991469396917269500	0.07991469396917 2	$0.000000e+000$	0.0017
0.100000	0.09983341664682814100	0.093197886168727	$2.775558e-017$	0.0022
0.120000	0.11971220728891936000	0.119712207288919	$2.775558e-017$	0.0025
0.140000	0.13954311464423647000	0.139543114644236	$8.326673e-017$	0.0028
0.160000	0.15931820661424598000	0.152733199752126	$8.326673e-017$	0.0032
0.180000	0.17902957342582418000	0.17902957342582	$1.942890e-016$	0.0034
0.200000	0.19866933079506119000	0.19866933079506	$3.608225e-016$	0.0038

Finally, Problem 5 is another nonlinear fourth-order problem which was recently solved by Duromola et al. (2024), the problem was resolved using the newly proposed method of order $P = 7$, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 11. It was also compared with exiting methods namely; Duromola (2024) who proposed a Chebyshev generated block method with six hybrid points, the method is of order of accuracy of $p = 7$. The comparison is shown in Table 12.

Table 12: Comparison of the newly proposed method with Duromola et al. (2024) for Problem 5.

x-value	Error in New Results, $P = 7$	Error in Duromola et al (2024), $P = 7$
0.02	$0.000000e + 000$	$2.4887e - 16$
0.04	$0.000000e + 000$	$3.1200e - 14$
0.10	$2.775558e - 017$	$1.7832e - 11$
0.16	$8.326673e - 017$	$4.4589e - 10$
0.18	$1.942890e - 016$	$9.9195e - 10$
0.20	$3.608225e - 016$	$2.0216e - 09$

5 Discussion of Results

Two non-linear ODEs were considered because of a small number of literatures that appeared on it. While other three problems ranging from linear to special and non-homogeneous fourth-order ordinary differential equations problems are examined by the new developed block methods. Problem 1 is a linear problem which was solving by Abolarin et al. (2020) who proposed a numerical model with $k = 2$ containing two-hybrid points with the presence of higher-derivative. The method is of order $p = 7$. Furthermore, the newly proposed block method $k = 3$ with fourhybrid points, which consist of many interpolation points. The numerical results is shown in Table 3. It could be seen that when the new method were compared with Abolarin et al. (2020) for solving Problem 1 in Table 4. In term of accuracy, the generated numerical results of the new block method $k = 3$ claim superiority over Abolarin et al. (2020) for solving Problem 1.

In Table 5, the results produced when the new method $k = 3$ was applied to Problem 2 are found better than the methods of: Ahamad and Charan (2019) and Jena et al. (2018). Ahamad and Charan (2019) employed Runge-kutta method to solve problem 2 after transforming to system of first order ordinary differential equation. On the other hand, Jena et al. (2018) presented a block algorithm with nine concurrent step with order $p = 11$. The results displayed in Table 5 for solving Problem 2 implies that the new block method $k = 3$ is high in accuracy than Jena et al. (2018) even with the same step-length $k = 9$. Problem 2 was also solved by the new block method $k = 3$ and the produced numerical results are better than the existing methods in the literature as shown in Table 5.

The numerical results of nonlinear problem is presented in Table 6, while In Tables 7, it can be observed that the accuracy of the new block method $k = 3$ for solving Problems 3 is higher than Kuboye et al. (2020), $k = 5$ despite having the same order $p = 7$. This actually shows that the method is computational reliable in handling nonlinear problem more efficient.

Problem 4 is a special fourth-order problem. The application of the new method to Problem 4 in Table 8. Table 9 is also better in terms of error than Kuboye et al (2020), $k = 5$, and Duromola (2016), $k = 1$. It should be noted that the two exiting methods compared with are also of order $p = 7$. Furthermore, by comparing errors in Tables 9 for solving Problem 4, it can be seen that the accuracy of the new block method $k = 3$ is more advanced than Kuboye et al (2020), $k = 5$, and Duromola et al. (2016).

In Table 10, the numerical results derived from the new block method $k = 3$ when the method was applied to Problem 5 (another nonlinear problem) show the efficiency of the method in terms of error over Duromola et al. (2024), $k = 6$ with multiple hybrid parameters. The accuracy of the new method in Table 11 is better when comparison was made with Duromola et al. (2024), $k = 6$, in spite of the higher step-length in the former method.

Finally, this research has demonstrated that block hybrid method with multiple interpolation points is more efficient and perform better than other block algorithms for solving fourth-order ordinary differential equation. This claim has been demonstrated with several examples presented ranging from linear, nonlinear, special and non-homogeneous problems. It should also be noted that most of the existing methods compared with are of order $p = 7$ or higher, despite the proposed method give a more accurate and efficient results. Hence it is recommended for general purpose.

Conclusion and Future Research

The proposed method which is design for the solution of fourth order initial value problems of ordinary differential equations with continuous hybrid block method of orders seven proposed for the direct solutions of fourth order initial value problems. The block methods are found to be zero-stable, consistency, hence they are convergent. five numerical examples were solved using the newly derived methods, comparing their accuracy with existing methods, they performed favorably well. The main method and the additional methods are obtained via interpolation and collocation procedures from the same continuous scheme derived. Numerical results obtained using the proposed block approach shows that it is adequate for the solution of special, linear, non-linear problems of fourth-order ordinary differential equations.

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