A QUEUING MODEL FOR CUSTOMERS' FLOW IN BANKING SECTOR IN RURAL AND URBAN CENTRES.

Etaga, H.O., Ajao, K., Awopeju, A., Etaga, M.C
[1]Department of Statistics, Nnamdi Azikiwe University,
Awka, Anambra State, Nigeria
[2]Department of Educational Foundations,
Nnamdi Azikiwe University, Awka, Anambra State, Nigeria
Email: ho.etaga@unizik.edu.ng

ABSTRACT

Queuing system or waiting line theory is primarily concerned with processes characterized by random arrivals (i.e., arrival at random time interval); the servicing of the customer is also a random process. Obtaining a good model for a queuing system requires an understanding of key components of the queuing system from which the system characteristics are derived. Flow of customers in a banking sector can be affected by location. The research is set to investigate possible variation in queuing model of banks with respect to geographical location. Two rural banks and two urban banks were selected. Queuing model were used of the data collected. The results showed that the queue models depend on location of the bank and banks cited in similar location have similar models.

Keyword: Banking Waiting Time, Queuing, Service time, Arrival time.

Introduction.

Queuing theory is the mathematical study of waiting lines of customers in a service system such as fuel stations, supermarket check-out counters, post offices, cafeteria, and banking halls. In queuing theory, a model is constructed so that important queuing characteristics of the service systems can be obtained as a measure of the service performance of the systems. Examples of such characteristics are queue lengths (number of customers waiting to be served), the waiting times involved etc.

Obtaining a good model for a queuing system requires an understanding of key components of the queuing system from which the system characteristics are derived. These components include:

- The arrival process which describes how customers appear for service in the system, based mainly on the arrival rate of customers per unit time and the distribution of the number of customer arrivals in the system per unit time.
- The service process which describes the number of servers, the service rate of customers per unit time and the service time distribution
- The queuing discipline which describes the order in which customers are served, for example First Come First Served (FCFS), Last Come First Served (LCFS) or Service By Appointment (SBA).

The queuing system therefore encompasses the arrival process, service process, the queuing discipline and assumptions about how the system works.

Queuing system or waiting line theory is primarily concerned with processes characterized by random arrivals (i.e., arrival at random time interval); the servicing of the customer is also a random process. Assuming that there are costs associated with waiting in line, and there are costs of adding more channels (i.e. adding more service facilities), it is possible to minimize the sum of the costs of waiting and the costs of providing service facilities. The computations will lead to such measures as the expected percentage utilization of the service facilities. These measures can then be used in the cost computation to determine the number and capacity of the service facilities that are desirable.

This consists of the arrival and service processes, the number of server and the assumptions regarding the service system. For example, it is assumed that customers come randomly into the banks, the service times for different server are assumed to be identically independent random variables, and the service times for different customers are independent of each other and also of the arrival process.

Queuing system is made up of arrival process, the service process, number of servers and queuing discipline. A general representation of the system is $A/\Delta/S$ for arrivals, services and number of servers.
A special set of queuing systems of usual interest in most studies is when the arrival and service time distributions are exponential or equivalently when the number of arrivals and completed services are Poisson distributed. Such a system with S servers is denoted by M/M/S, with M representing the exponential inter-arrival and service times with S servers.
More generally, a queuing system consists of G/G/S in which G denotes general arrival and or service times. Hence, a suitable queuing model for describing the Banks deposit customers' experiences could be M/M/S, M/G/S, G/M/S, or G/G/S.

Modern day banks in Nigeria are designed not to have queues by the deployment of various information technology platforms. While most banks have become virtually queue free, many others, including different branches of the same bank(s) have continued to have customers waiting for longer period before service, especially during money deposition

LITERATURE REVIEW

Harley et al (2014) investigated the impact of various elements of customer services adopted by some Nigerian banks to improve bank profitability in the Nigerian banking industry, and found that the average time a bank customer spends waiting in the queue to carryout banking transaction had a linear relationship with the bank profitability. The study established that there is an inverse relationship between banks customer services and profitability in Nigeria banks.

Jacob and Szyszkowski (2009) investigated the queuing procedure at a call centre, and ascertained that depending on the nature of data, the abandoning time is general and independently distributed, different distributions may be used for different sets of call centre data. The service times of the used location was observed to follow a Poisson distribution, and was the best model for call centers.

Emeka and Favour (2012) worked on Automated Teller Machine (ATM) utilization in eight locations of four banks in Ibadan metropolis, Nigeria, and found that more female bank customers patronize the ATMs than male customers. More relatively younger customers used ATMs than the adults, more customers with relatively higher level of education used ATMs than the illiterates, and more student customers agreed that the machines had benefited them more than other groups.
Constantinos and Mieghem (2005) studied how multi-product queuing systems should be controlled so that sojourn times (or end-to-end delays) do not exceed specified lead times. The

main benefit of the approach, according to the researchers is that it is possible (and relatively easy) to construct scheduling and multi-product admission policies for lead time control which is simpler than a heavy-traffic approach. The admission policies that emerged from it are also more specific and consistent. Therefore, it was concluded that M/M/S is the best model for the organization.

According to Toshiba et al (2013), lines of waiting customers are always very long in most of the bank. In their work, they converted the M/M/S/∞ or FCFS model into M/M/1/∞ or FCFS in order to find out which is more efficient, a line or more lines. Based on this work the result of analysis was effective and practical. Also, the time of customer queuing was found to be reduced while customer satisfaction increased. Therefore, the best model for the case study is M/M/1/∞.

Adeleke et al (2009) considered the waiting of patients in a University health centre as a single–channel queuing system with Poisson arrivals and exponential service rate where arrivals are handled on a first come first serve basis. They observed that the traffic intensity; p = 0.8444, is the probability of patients queuing on arrival. This clearly indicates a higher possibility of patients waiting for treatment since the doctor maybe busy rendering service to a patient that has earlier arrived.

Patel and Bhathawala (2012) worked on time wastage in the use of Automated Teller Machine (ATM), and found out that the arrival rate at a bank's ATM on Mondays during banking time was 1 customer per minute while the service rate was 1minute 26seconds. The probability of buffer overflow was the probability that, customers will wait for so long in the queue.
Nafees (2007) worked on analysis of queuing system for the empirical data of supermarket checkout service. He observed that the model used contains five servers which are checkout sales counters; attached to each server is a queue (M/M/5). In any service system, a queue forms whenever current demand exceeds the existing capacity to serve. This occurs when the checkout operation unit is too busy to serve the arriving customers immediately. They concluded that the model be increased to accommodate more servers in the system to reduce traffic intensity.

Ohaneme et al (2012) used petrol service stations as an avenue to assess the importance of queuing in a system, and observed that the service points sell products randomly to available customers which caused long queues in the service points thereby increasing customers' waiting time. Also, the results obtained show that there is a tremendous improvement in the efficiency of the customer services when the queuing system of M/M/6 is strictly adhered to. This ensures that the average waiting time of the customer is drastically reduced.

Chandra and Madhu (2013) used queuing with a Markovian queuing system having a multi-task service counters and finite queue in front of each counter. Total service of a customer is completed in three stages provided by two servers at three counters. The first server (S1) can serve the counter I and III alternatively, whereas second server (S2) provides the service at counter II. The steady state queue size distribution was obtained. The analysis was carried out to study the effect of variation of different parameters. The researchers concluded that state dependent rate incorporated for modeling multi-counters system makes the results closer to realistic situation. This implies that, the use of blocking in modeling of queuing system makes it more reliable than any other method. However, they unanimously agreed that the work can be extended by considering bulk arrivals and/or bulk service in which direction the attention should be paid.

Disney (1981) examined the internal arrival rate distribution with feedback flow as a generalization of Jackson's model. His research showed that when a system has any kind of feedback flow, the internal flows in the system do not follow the Poisson distribution. Thus, the assumption of Poisson arrival is justified only when the system under consideration has either a tandem or arbitrarily linked network configuration with feed-forward flows. It is, however, known that the Jackson's product-form solution holds regardless of whether or not internal flows are Poisson.

Melamed (1979) extended Burke's finding in an open Jackson system. He showed that departure rate from internal stations to outside the system are mutually independent if arriving rates to all internal stations are Poisson distribution. The finding implies that the sum of all departure rates from the network must also be Poisson.

Hunt (1956) examined a finite buffer between two stations. He used a single server with a sequential two-station model to compare traffic intensities for the three basic cases. These cases included where an infinite buffer exists between the stations; and where a finite station may have a finite buffer; where zero buffer exists between the stations; and where a finite buffer (>0) exists between the station, with the exception that the first station may have an infinite buffer. The model is based on Markov chains and thus has Poisson arrival rates and exponential service times.

The first single –mode decomposition methodology was introduced by Hiller and Boling (1967). Their study used an open system with a tandem configuration where every station is equipped with multiple servers. The service time in their model was assumed to follow an exponential distribution, while the arrival process was assumed to be Poisson. With the exception of the first station, the arrival time was recorded at the time of the service completion at the previous station and the departure time was recorded at the time of physical departure from the station. The model assumed that the first station was considered saturated, meaning that the first station was constantly full. This implies that the arriving rate to the second station is always to the service rate at the first station.

Takahashi et al (1980) developed a single-node decomposition approximation method to solve an open queuing network model with feed-forward flows. Their model assumed Poisson arrivals exponential service times, and a single server with finite buffer stations.

METHODOLOGY

Non-probability sampling technique (convenient sampling) was used in the selection of banks for the study as purposive sampling was carried out to select rural area with banks whose sub-branch also exist in the urban center of the State. Abagana has First Bank branch Plc and it is also at Awka which is urban centre. In the same locality, Union Bank branch Plc has a branch at Abagana and Awka.

Therefore, Abagana qualifies to be considered in the study as a rural area that is location with sparse (low) population. Also Awka the State capital of Anambra, is an important urban centre in the State.

The data used for the study is a primary data as they were collected using direct observations. For accurate data collection, four (4) observers and one supervisor was used. Two observers recorded customers' arrival time while the other recorded service begin and service end for the customers.

Test Statistic

The test statistic used for hypothesis testing include Chi-square Goodness-of-Fit for test of suitability of distribution

Chi-square test statistic is given by $\chi^2 = \sum_{i=1}^{m} \left(\frac{(o_i - e_i)^2}{e_i}\right)$ 1

Where o_i = the observed frequency, e_i= the expected frequency of the classes. m = is the effective number of classes

QUEUING MODEL

Model Specification: If the analyses of the queuing data show that the arrival time and service time follow exponential distribution with known number of servers, M/M/S model is appropriate for the study. Otherwise, an alternative model will be adopted such as;

(i) M/G/S
(ii) (ii) G/M/S and
(iii) (iii) G/G/S

In the model (i), arrival time is exponentially distributed, Service time not specified and the capacity is known. In the model (ii), arrival time is not specified, Service time exponentially distributed and the capacity is known. In the model (iii), arrival time is not specified, Service time is not specified and the capacity is known.

For multiple servers, s, some characteristics of the model are calculated thus:

$$Mean\ Arrival\ Rate\ (\lambda) = \frac{Total\ Arrival\ Time}{Number\ of\ customers} \quad 2$$

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ customers} \quad 3$$

$$Mean\ Waiting\ Time = \frac{Total\ Waiting\ Time}{Number\ of\ Customers} \quad 4$$

Traffic intensity $\rho = \frac{\lambda}{s\mu}$ where λ = the mean arrival rate, μ = the mean service rate, s = the number of service points.

5

The probability of having exactly zero number of customers in the system or probability that the system is idle is P_0 which is obtained as

$$P_0 = \left[\sum_{n=0}^{S} \frac{1}{n!}(s\rho)^n + \sum_{n=s+1}^{1} \frac{s^s}{s!}\frac{\rho^n}{1-\rho}\right]^{-1}$$

$$= \left[\sum_{n=0}^{S} \frac{1}{n!}(s\rho)^n + \frac{S^S}{S!}\frac{\rho^{S+1}}{1-\rho}\right]^{-1}$$

For s = 3

$$P_0 = \left[1 + 3\rho + \frac{(3\rho)^2}{2} + \frac{1}{6}(3\rho)^3 + \frac{3^3}{3\times 2}\frac{\rho^4}{1-\rho}\right]^{-1} \qquad 6$$

The probability of having servers in the system is given by

$$P_s = \frac{1}{n!}(s\rho)^n P_0 \qquad 7$$

where s is number of servers and n is number of customers attended to simultaneously

The probability that all servers are busy is obtained from equation 3.4 on the waiting time distribution of such a model

$$\text{P}[W(t) \geq y] = P_s(1-\rho)^{-1}\rho^{-\mu s(1-\rho)y}; \quad y \geq 0 \qquad 8$$

where$W$(t) represents waiting time, therefore all servers are busy when $W(t) \geq 0$ with probability $= \frac{P_s}{1-\rho}$ 9

The expected number of people waiting to be served is given by

$$\text{E(N)} = \frac{\rho P_0}{(1-\rho)^2} \qquad 10$$

The expected time that customer waits for service

$$\text{E (W(t))} = \frac{P_s}{\mu S(1-\rho)^2} \qquad 11$$

If a customer has to wait, the expected length of his waiting time $= \frac{1}{\mu s(1-\rho)}$ 3.12

Probability that a customer will queue on arrival = $\left(\frac{(\rho s)^s}{s!(1-\rho)}\right) P_0$ 13

Probability of not queuing on arrival is = 1 — $\frac{(\rho s)^s}{s!(1-\rho)} P_0$ 14

RESULTS:

Location 1: (Rural Area)

This bank has 3 servers. From the data collected during field survey the following results were obtained. This bank has 3 servers.

Mean Arrival Rate

$$Mean\ Arrival\ Rate\ (\lambda) = \frac{Total\ Arrival\ Time}{Number\ of\ customers} = \frac{487}{456} = 1.0679$$

Similar computation was done for the other days used for field survey and we have;

Table 1: Mean Arrival Rate of Customers in Rural Area of Union Bank

Day 1	Day 2	Day 3	Day 4	Day 5	Average
1.0679 min.	1.6333min.	2.1859min.	1.9123min.	1.8458min.	1.8280min.

From the results in Table 1, the mean arrival rate of customers in the rural area of the branch of Union bank for the first day is 1.1minutes and 1.8minutes for the last day of the week. On the average, the mean arrival rate within the week is 1.8minutes.

Computation of Mean Service Rate

Mathematically, the Mean Service Rate (MSR) can be computed using the expression

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ \ customers}$$

Based on the data collected, the mean service time of the branch of the bank by days of the week is $Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ \ customers} = \frac{238}{98} = 2.3878$

Table 2: Mean Service Rate of Customers in Rural Area of Union Bank

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Server 1	2.3878min.	2.3778min.	2.3390min.	2.3158min.	2.2250min.	2.3291min.
Server 2	2.3061min.	2.1332min.	2.1864min.	2.2895min.	2.2250min.	2.2280min.
Server 3	2.1939min.	2.1111min.	2.1186min.	2.2632min.	2.1625min.	2.1698min.
General Mean	2.2959min.	2.2074min.	2.2147min.	2.2895min.	2.2042min.	Grand Mean = 2.2423min.

The mean service rate of customers in the first day is 2.3minutes and on average is 2.2423minutes within the week.

Computation of Mean waiting time

Mean Waiting Time can be computed using the expression

$$Mean\ waiting\ Time = \frac{Total\ waiting\ Time}{Number\ of\ \ customers}$$

Based on the data collected, the mean waiting time of the branch of the bank for the first day of the week is; $Mean\ waiting\ Time = \frac{Total\ waiting\ Time}{Number\ of\ \ customers} = \frac{38}{98} = 0.3878\,\text{min}$.

Table 3: Mean waiting Time of Customers in Rural Area of Union Bank

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Server 1	0.3878min.	0.4444min.	0.0169min.	0.0658min.	0.0750min.	0.1979min.
Server 2	0.3571min.	0.3778min.	0.0339min.	0.0263min.	0.0625min.	0.1715min.
Server 3	0.3061min.	0.3111min.	0.0000min.	0.0395min.	0.0103min.	0.1334min.
General Mean	0.3503min.	0.3778min.	0.0169 min	0.0439min.	0.0802min.	Grand Mean = 0.1676min.

From the result in Table 3, server 1 has the highest waiting time and server 3 has the least waiting time. Using the average waiting time, it can be concluded that server 3 is the most efficient server among all, since time spent on the queue by the customers is at the minimal level.

Computation of Traffic intensity

Traffic intensity (ρ) can be computed using the expression

$$\rho = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(Number\ of\ \ Servers(S) \times Mean\ Service\ Rate\ (\mu))} = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(3 \times Mean\ Service\ Rate\ (\mu))}$$

To compute traffic intensity for the bank, Tables 4.1 and 4.2 were used.

Table 4: Traffic Intensity

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
λ	1.4728min.	1.6333min.	2.1859min.	1.9123min.	1.8458min.	1.8280min.
S. μ	6.8877min	6.6222min	6.6441min	6.8685min	6.6126min	6.7269min
ρ	0.2138min	0.2466min.	0.3290min.	0.2784min.	0.2791min.	0.2717min.

Traffic Intensity is a measure of the average occupancy of a server or resource during a specified period of time, normally a busy hour. Therefore, from the computation, the intensity was at the peak on the third day and the least on the first day of the week.

Computation of Probability of Idleness of the system (P_0)

The probability of having exactly zero number of customers in the system or probability that the system is idle is P_0 which is obtained as

$$P_0 = \left[1 + 3\rho + \frac{(3\rho)^2}{2} + \frac{1}{6}(3\rho)^3 + \frac{27\rho^4}{6(1-\rho)}\right]^{-1}$$

For day 1; $\rho = 0.2138$ (*see Table* 4.4)

$$P_0 = \left[1 + 3(0.2138) + \frac{9(0.2138)^2}{2} + \frac{27}{6}(0.2138)^3 + \frac{27(0.2138)^4}{6(1-0.2138)}\right]^{-1}$$

$$P_0 = [1 + 0.6414 + 0.2057 + 0.04397 + 0.001195]^{-1} = 0.5255$$

Similar computation was done for all the five days. See Table 4.5.

Table 5: Probability of Idleness of System and Traffic Intensity

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2138min	0.2466min.	0.3290min.	0.2784min.	0.2791min.	0.2717min.
P_o	0.5255	0.4755	0.3686	0.4313	0.4304	0.4403

The higher the probability of idleness, the greater the possibility of idleness of a system. From Table 5, it can be seen that the system has highest possibility of been idle in the first day and has the least possibility of idleness on the third day.

Computation of Probability of having Servers in the System

The bank has three servers. Therefore, there is need for P_1, P_2 and P_3.
The probability of having servers in the system is given by

$$P_s = \frac{1}{n!}(s\rho)^n P_0$$

where s is number of servers and n (numbers of customers attended to simultaneously) = 3

Table 6: Probability of having Server in the System

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.2138min	0.2466min.	0.3290min.	0.2784min.	0.2791min.	0.2717min.
P_o	0.5255	0.4755	0.3686	0.4313	0.4304	0.4403
P_s	0.0231	0.0321	0.0591	0.0419	0.0421	0.0397

The higher the probability of an event, the higher the chance of occurrence. As observed in Table 6, the probability of having servers in the system is at the peak on the third day.

7. Computation of Probability of servers been busy. P(W(t))

Busy server is necessary in the process as idleness implies redundancy which can be interpreted as wastage. Mathematically, probability of server been busy can be computed using equation 8; It can also be computed using 1 minus probability of idleness of the system. Table 7 is computed from Table 5.

Table 7: Probability of servers been busy

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.2138min	0.2466min.	0.3290min.	0.2784min.	0.2791min.	0.2717min.
P_o	0.5255	0.4755	0.3686	0.4313	0.4304	0.4403
$(1 - P_o)$	0.4745	0.5245	0.6314	0.5687	0.5696	0.5597

Table 7 shows that servers are busier on the third day than any other day and on the average, the servers are busy 56% of the working hours.

Computation of Expected Numbers of People Waiting to be Served E(N)

The expected number of people waiting to be served as given in equation 7

Table 8: The Expected Number of People Waiting To Be Served

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.2138	0.2466	0.3290	0.2784	0.2791	0.2717
P_o	0.5255	0.4755	0.3686	0.4313	0.4304	0.4403
$(1 - \rho)$	0.7862	0.7534	0.671	0.7216	0.7209	0.7283
E(N)	0.1818	0.2066	0.2693	0.2306	0.2311	0.2255

From Table 8, on the first day at least 18 per cent of the available customers are expected to wait on queue. 21 per cent, 27 per cent, 23 per cent and 23 per cent for second, third, fourth and fifth day respectively. On the average, for rural area branch of the bank considered, it is expected that 23% of the customers waited on queue to be served.

Computation of Expected Time a Customer Waits for Service E(W(t))
The expected duration of waiting time of customer as given in equation 8;

For day 1, E(W(t)) = $\frac{0.0231}{(2.2959)3(0.7862)^2} = 0.0054$

Table 9: Expected Time a Customer waits for Service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P_s	0.0231	0.0321	0.0591	0.0419	0.0421	0.0397
μ	2.2959	2.2074	2.2147	2.2895	2.2042	2.2423
$(1 - \rho)^2$	0.6181	0.5676	0.4502	0.5207	0.5197	0.5304
E(w(t)) in mins	0.0054	0.0085	0.0198	0.0117	0.0123	0.0111
E(w(t)) in secs	0.3256	0.5124	1.1855	0.7029	0.7350	0.6676

As observed in Table 9, customers spent more time waiting for service on the third day than any other day as the expected waiting time for the day is 1second. The least waiting time was observed on the first day with waiting time of less than 1 second.

Computation of Conditional Probability of waiting time for service
If a customer has to wait, the expected length of his waiting time, given in equation 12. Using the collected data, the results of the computation are as shown in Table 10

Table 10: Conditional Probability of waiting time for service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
μ	2.2959	2.2074	2.2147	2.2895	2.2042	2.2423
$(1 - \rho)$	0.7862	0.7534	0.671	0.7216	0.7209	0.7283
$E[W(t) \mid W(t) \geq 0]$	0.1847	0.2004	0.2243	0.2018	0.2098	0.2041

Hint: s = 3.
The expected duration of customer waiting for service on the first day if at all there is queue is 0.1847min and on the average, the customer waiting time is 0.2041min.

Computation of Probability that a customer will queue on arrival
This aspect is different from expected waiting time as it shows the possibility of a customer on arrival waiting to be served. The higher the probability of a customer queuing on arrival, the longer the queue in the system. Mathematically, this can be computed using the expression in equation10:

For day 1, rho is 0.2138, P_0 is 0.5255 and s = 3. Therefore, the probability of customer queuing on arrival is $\left(\frac{(0.2138*3)^3}{3!(1-0.2138)}\right)0.5255=0.0294$
Similar computation was done for other days. See Table 11.

Table 11: Probability that a customer will queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2138	0.2466	0.3290	0.2784	0.2791	0.2717
P_o	0.5255	0.4755	0.3686	0.4313	0.4304	0.4403
P(Queuing)	0.0294	0.0426	0.0868	0.0580	0.0584	0.0337

From Table 11, day 3 has the highest probability of customer waiting on arrival before service.

Probability of not queuing on arrival

This can be computed using the expression; 1 minus probability of queuing on arrival. Then, we have;

Table 12: Probability that a customer will not queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P(Queuing)	0.0294	0.0426	0.0868	0.0580	0.0584	0.0337
P(Not Queuing)	0.9706	0.9574	0.9132	0.9420	0.9416	0.9663

Test of goodness of fit using chi-square

This is used to test whether arrival time and service time follow exponential distribution.

Table 13: Mean Arrival Time and Service Time

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Arrival Time	1.4728 min.	1.6333min	2.1859min.	1.9123min	1.8458min	1.8280min.
Service Time	2.2959min	2.2074min	2.2147min	2.2895min	2.2042min	2.2423min

Chi-Square Test of Goodness-of-fit of Arrival Time

To test the hypothesis;

H_0: the data follow exponential distribution. vs

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 14: Observed and Expected for the Goodness-of-fit

Observed [O(x)]	**P(x)**	**Expected [E(x)]**	**(Obs– Exp.)**	$(\mathbf{Obs} - \mathbf{Exp})^2$	**Chi-Sq. Value**
1.4728	0.1663	1.8091	-0.3363	0.11309769	0.062516
1.6333	0.1374	1.374	0.2593	0.06723649	0.048935
2.1859	0.06988	0.6988	1.4871	2.21146641	3.164663
1.9123	0.9737	1.0592	0.8531	0.72777961	0.687103
1.8458	0.1056	1.1486	0.6972	0.48608784	0.4232

From the computation, $\chi^2_{calculated}$ is4.39.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Conclusion: Comparing both calculated and tabulated values of Chi-Square; 4.39 and 9.49, there exists enough evidence to accept the null hypothesis and conclude that the data follow Exponential distribution.

Chi-Square Test of Goodness-of-fit of Service Time
To test the hypothesis:
H_0: the data follow exponential distribution.
H_1; the data do not follow exponential distribution.
Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 15: Observed and Expected for the Goodness-of-fit

Observed	P(x)	Expected	Obs–Exp	$(\mathbf{Obs - Exp})^2$	Chi-Sq. Value
2.2959	0.037099	0.4991	1.7968	3.22849024	6.468624
2.2074	0.04212	0.4719	1.7355	3.01196025	6.382624
2.2147	0.04168	0.5667	1.648	2.715904	4.79249
2.2895	0.03744	0.5037	1.7858	3.18908164	6.331312
2.2042	0.04232	0.5693	1.6349	2.67289801	4.695061

Chi-Square value is 28.67.
Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Conclusion: Comparing both calculated and tabulated values of Chi-Square; 28.67 and 9.49, there is enough evidence to reject the null hypothesis and conclude that the data do not follow Exponential distribution. Since the data do not follow exponential distribution, this implies that for the branch of the bank, the service time follows general arrival (G).
The queuing model for rural branch of Union Bank; Abagana branch, is M/G/S.

Location 2: Union Bank (Urban Area)
The bank is situated at Zik's Avenue, Awka. Based on the data collected during field survey (See Appendix 2), we have;
The bank has 3 servers.

Mean Arrival Rate

$$Mean\ Arrival\ Rate\ (\lambda) = \frac{Total\ Arrival\ Time}{Number\ of\ customers} = \frac{433}{390} = 1.1103$$

Similar computation was done for the other days used for field survey and we have;

Table 16: Mean Arrival Rate (Minutes) of Customers in Urban Area of Union Bank

Day 1	Day 2	Day 3	Day 4	Day 5	Average
1.1103	1.6333	2.5819	1.1123	1.8458	1.8280

From the Table 16, the result shows that the mean arrival rate of customers in the urban area branch of Union bank for the first day is 1.1minute and 1.8minutes for the last day of the week.

On the average, the mean arrival rate within the week is 1.8 minutes. The highest arrival rate was recorded in the third day of the week with mean arrival rate of 2.6minutes.

Computation of Mean Service Rate

Mathematically, the Mean Service Rate (MSR) can be computed using the expression

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ customers}$$

Based on the data collected, the mean service time for server 1 of the branch of the bank for the first day of the week is

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ customers} = \frac{287}{130} = 2.2077$$

Similar computation was done for all the servers for number of day used during the field survey. See Table 17.

Table 17: Mean Service Rate of Customers in Urban Area of Union Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1 (Min.)	2.2077	2.8235	2.7788	2.2523	2.3290	2.4650
Server 2 (Min.)	2.1615	2.7479	2.6903	2.2523	2.3097	2.4204
Server 3 (Min.)	2.1615	2.7479	2.6637	2.1441	2.2968	2.3933
General Mean(Min.)	2.1769	2.7731	2.7189	2.2162	2.3118	2.5211

The day 2 recorded the highest mean service rate with 2.8minutes and on average 2.5minutes within the week.

Computation of Mean waiting time

Mean Waiting Time (MWT) can be computed using the expression

$$Mean\ WaitingTime = \frac{Total\ Waiting\ Time}{Number\ of\ customers}$$

Based on the data collected, the mean waiting time of the branch of the bank for the first day of the week is

$$Mean\ Waiting\ Time = \frac{Total\ Waiting\ Time}{Number\ of\ customers} = \frac{75}{130} = 0.5769\min .$$

Table 18: Mean Waiting Time of Customers in Urban Area of Union Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1	0.5769	1.7143	0.8673	0.1802	2.6323	1.1942
Server 2	0.5154	1.7983	0.8673	0.0721	2.7290	1.1964
Server 3	0.4846	1.6923	0.8407	0.0991	2.8129	1.1859
General Mean	0.5256	1.7352	0.8584	0.1171	2.7247	1.1922

From the result in Table 18, on the average, server 2 has the maximum waiting time and server 3 has the least waiting time. Using the average waiting time, it can be concluded that server 3 is the most efficient server among all, since time spent on the queue by the customers is at minimal level.

Computation of Traffic intensity

Traffic intensity (ρ) can be computed using the expression

$$\rho = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(Number\ of\ \ Servers(S) \times Mean\ Service\ Rate\ (\mu))} = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(3 \times Mean\ Service\ Rate\ (\mu))}$$

To compute traffic intensity for the bank, Tables 4.16 and 4.17 were used.

Table 19: Traffic Intensity in Urban Centre of Union Bank Branch

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
λ (min.)	1.4727	1.6333	2.5819	1.1123	1.8458	1.8280
S. μ	6.5307	8.3193	8.1567	6.6486	6.9354	7.5633
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417

Traffic Intensity is a measure of the average occupancy of a server or resource during a specified period of time, normally a busy hour. Therefore, computation shows that, the intensity was at the peak on the third day and the least on the fourth day of the week.

5. Computation of Probability of Idleness of the system (P_0)

The probability of having exactly zero number of customers in the system or probability that the System is idle is P_0 which is obtained as

$$P_0 = \left[1 + 3\rho + \frac{(3\rho)^2}{2} + \frac{1}{6}(3\rho)^3 + \frac{27\rho^4}{6(1-\rho)}\right]^{-1}$$

For day 1; $\rho = 0.2255$

$$P_0 = \left[1 + 3(0.2255) + \frac{9(0.2255)^2}{2} + \frac{27}{6}(0.2255)^3 + \frac{27(0.2255)^4}{6(1-0.2255)}\right]^{-1}$$

$$P_0 = [1 + 0.6765 + 0.2288 + 0.0516 + 0.0150]^{-1} = 0.5071$$

Similar computation was done for all the 5 days. See Table 4.20.

Table 20: Probability of Idleness of System and Traffic Intensity

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417
P_o	0.5071	0.5541	0.3833	0.6049	0.4479	0.4827

The higher the probability of idleness, the greater the possibility of idleness of a system. From Table 20, the analysis shows that the system has highest possibility of been idle in the fourth day and has the least possibility of idleness on the third day.

Computation of Probability of having Servers in the System

The probability of having servers in the system is computed using equation 7

Table 21: Probability of having Server in the System

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417
P_o	0.5071	0.5541	0.3833	0.6049	0.4479	0.4827
P_s	0.0262	0.0189	0.0549	0.0127	0.0380	0.0307

The higher the probability of an event, the higher the chance of occurrence, as observed in Table 21, the probability of having servers in the system is at the peak on the third day.

Computation of Probability of servers been busy. P(W(t))

Busy server is necessary in the process as idleness implies redundancy which can be interpreted as wastage. Mathematically, probability of server been busy can be computed using the equation 3.8

It can also be computed using 1 minus probability of idleness of the system. Table 22 is computed from Table 20.

Table 22: Probability of servers been busy

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417
P_o	0.5071	0.5541	0.3833	0.6049	0.4479	0.4827
$(1 - P_o)$	0.4929	0.4459	0.6167	0.3951	0.5521	0.5173

Table 22 shows that in the location servers are busier on the third day than any other day and on the average, the servers are busy 52% of the working hours.

Computation of Expected numbers of people waiting to be served

The expected number of people waiting to be served is given in equation 10

Table 23: Expected numbers of people waiting to be served

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417
P_o	0.5071	0.5541	0.3833	0.6049	0.4479	0.4827
$(1 - \rho)^2$	0.5999	0.6459	0.4672	0.6934	0.5386	0.5750
E(N)	0.1906	0.1684	0.2597	0.1459	0.2213	0.2029

From Table 23, on the first day at least 19 per cent of the available customers are expected to wait on queue. 16 per cent, 25 per cent, 14 per cent and 22 per cent for second, third, fourth and fifth day respectively. On the average, for urban area branch of the bank considered, it is expected that 20% of the customers waited on queue to be served.

Computation of Expected time a customer waits for service E(W(t))

The expected duration of waiting time of customer as given in equation 11 is computed

For day 1, $E(W(t)) = \frac{0.0231}{(2.2959)3(0.7862)^2} = 0.0054$

Table 24: Expected Time a Customer waits for Service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P_s	0.0262	0.0189	0.0549	0.0127	0.0380	0.0307
μ	2.1769	2.7731	2.7189	2.2162	2.3118	2.5211
$(1 - \rho)^2$	0.5999	0.6459	0.4672	0.6934	0.5386	0.5750
E(w(t)) in mins	0.0067	0.0035	0.0144	0.0028	0.0102	0.0071
E(w(t)) in secs	0.4020	0.2100	0.8640	0.1680	0.6120	0.4260

As observed in Table 24, customers spent more time waiting for service on the third day than any other day as the expected waiting time for the day is highest. The least waiting time was observed on the fourth day with waiting time of 0.1680minutes.

Computation of Conditional Probability of waiting time for service

If a customer has to wait, the expected length of his waiting time as given in equation 12, the result of the computation was shown in Table 25

Table 25: Conditional Probability of waiting time for service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
μ	2.1769	2.7731	2.7189	2.2162	2.3118	2.5211
$(1 - \rho)$	0.7745	0.8037	0.6835	0.8327	0.7339	0.7583
$E[W(t) \mid W(t) \geq 0]$	0.1977	0.1496	0.1794	0.1806	0.1965	0.1744

Hint: s = 3.

The expected duration of customer queue for service on the first day if at all there is waiting is 0.1977mins and on the average, the customer waiting time is 0.1744mins.

11. Computation of Probability that a customer will queue on arrival

This aspect is different from expected waiting time as it shows the possibility of a customer on arrival waiting to be served. The higher the probability of a customer queuing on arrival, the longer the queue in the system. Mathematically, this can be computed using the expression 13 For day 1, rho is 0.2255, P_0 is 0.5071 and s = 3. Therefore, the probability of customer queuing on arrival is $\left(\frac{(0.2255*3)^3}{3!(1-0.2255)}\right) 0.5071$ = 0.0338

Similar computation was done for other days. See Table 26.

Table 26: Probability that a customer will queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2255	0.1963	0.3165	0.1673	0.2661	0.2417
P_o	0.5071	0.5541	0.3833	0.6049	0.4479	0.4827
P(Queuing)	0.0338	0.0238	0.0800	0.0153	0.0517	0.0404

The result shown in Table 26 that, day 3 has the highest probability of customer waiting on arrival before service.

Probability of not queuing on arrival

This can be computed using the expression; 1 minus probability of queuing on arrival. Then, we have;

Table 27: Probability that a customer will not queue on arrival

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
P(Queuing)	0.0338	0.0238	0.0800	0.0153	0.0517	0.0404
P(Not Queuing)	0.9662	0.9762	0.9200	0.9847	0.9483	0.9596

Test of goodness of fit using chi-square.

This is used to test whether arrival time and service time follow exponential.

Table 28: Mean Arrival Time and Service Time

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Arrival Time	1.4727	1.6333	2.5819	1.1123	1.8458	1.8280
Service Time	2.1769	2.7731	2.7189	2.2162	2.3118	2.5211

A. Test of Goodness-of-fit of Arrival Time Using Chi-Square

To test the hypothesis

H_0: the data follow exponential distribution.

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 29: Observed and Expected for the Goodness-of-fit

Observed	**P(x)**	**Expected**	**Obs–Exp**	$(\mathbf{Obs - Exp})^2$	**Chi-Sq. Value**
1.4727	0.173706794	1.819404961	-0.346704961	0.12020433	0.066067936
1.6333	0.143521763	1.50324695	0.13005305	0.016913796	0.011251509
2.5819	0.046480014	0.486831668	2.095068332	4.389311314	9.016075985
1.1123	0.266593329	2.792298532	-1.679998532	2.822395066	1.010778409
1.8458	0.111488174	1.167727137	0.678072863	0.459782808	0.393741648

From the computation, the $\chi^2_{Calculated}$ is10.49.

$\chi^2_{Tabulated}$ is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Conclusion: Comparing both calculated and tabulated values of Chi-Square; 10.49 and 9.49, there exists enough evidence to reject the null hypothesis and conclude that the data do not follow exponential distribution. Since the data do not follow exponential distribution, this implies that for the branch of the bank, the arrival time follows general arrival (G).

A. Test of Goodness-of-fit of Service Time Using Chi-Square

To test the hypothesis

H_0: the data follow exponential distribution.

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 30: Observed and Expected for the Goodness-of-fit

Observed	P(x)	Expected	Obs–Exp	$(\mathbf{Obs - Exp})^2$	Chi-Sq. Value
2.1769	0.033823028	0.497807322	1.679092678	2.819352222	5.663541089
2.7731	0.013378257	0.196901183	2.576198817	6.636800342	33.70624912
2.7189	0.014555224	0.214223784	2.504676216	6.273402948	29.28434386
2.2162	0.031817062	0.468283511	1.747916489	3.055212051	6.524278511
2.3118	0.027420127	0.40356943	1.90823057	3.641343907	9.022843734

$\chi^2_{Calculated}$ is84.20.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

The analysis shows that the Chi-Square calculated value is greater than Chi-Square tabulated value then, there is enough evidence to reject the null hypothesis and conclude that the data do not follow Exponential distribution. Since the data do not follow exponential distribution, this implies for the branch of the bank, the service time follows general arrival (G).

In general, the queuing model for urban branch of Union Bank; Abagana branch, is G/G/S which implies the arrival time and service time are general and the number of server is known (3)

Location 3: First Bank (Rural Area)

The bank is also situated at Abagana, Old Enugu-Onitsha road. The bank has 3 servers. From the data collected during field survey see appendix 3 the analysis were computed thus

Mean Arrival Rate

$$Mean\ Arrival\ Rate\ (\lambda) = \frac{Total\ Arrival\ Time}{Number\ of\ \ customers} = \frac{535}{348} = 1.5374$$

Similar computation was done for the other days used for field survey and we have;

Table 31: Mean Arrival Rate of Customers in Rural Area of First Bank

Day 1	Day 2	Day 3	Day 4	Day 5	Average
1.5374 min.	1.3463min.	1.7463min.	1.6437min.	1.3242min.	1.5196min.

Considering the location and the branch, the mean arrival rate of customers of First bank in rural area was at its peak on the third day of the week with average arrival rate of 1.7minutes. The lowest arrival rate was recorded on day 5

.

Computation of Mean Service Rate

Mean Service Rate (MSR) can be computed using the expression;

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ \ customers}$$

The mean service time of the branch of the bank for the first day of the week is

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ \ customers} = \frac{436}{348} = 1.2529$$

Table 32: Mean Service Rate of Customers in Rural Area of First Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1	1.2529min.	1.8432min.	2.0431min.	2.1043min.	2.1309min.	1.8729min.
Server 2	1.4632min.	1.4493min.	1.5332min.	1.4937min.	2.1143min.	1.6107min.
Server 3	2.0432min.	1.8434min.	1.8474min.	1.3938min.	1.7362min.	1.7728min.
General Mean	1.5832min.	1.7265min.	1.8079min.	1.6639min.	1.9938min.	**Grand Mean** = 1.7521min.

Table 32 result shows that, the mean service rate of customers in the first day is 1.6minutes. On the average, it is 1.8minutes.

Computation of Mean waiting time

Mean Waiting Time (MWT) can be computed using the expression

$$Mean\ Waiting Time = \frac{Total\ Waiting\ Time}{Number\ of\ \ customers}$$

Based on the data collected, the mean waiting time of the branch of the bank for the first day of the week is

$$Mean\ Waiting\ Time = \frac{Total\ Waiting\ Time}{Number\ of\ \ customers} = \frac{396}{348} = 1.1379\min.$$

Table 33: Mean Waiting Time of Customers in Rural Area of First Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1	1.1379min.	1.2165min.	1.4430min.	1.4342min.	1.5932min.	1.3649min.
Server 2	1.2436min.	1.6531min.	1.2645min.	1.5427min.	1.5427min.	1.4493min.
Server 3	1.5436min.	1.5421min.	0.5842min.	1.5426min.	1.6326min.	1.3690min.
General Mean	1.3083min.	1.4706min.	1.0972min	1.5065min.	1.5895min.	**Grand Mean** = 1.3944min.

From the result in Table 33, on the average, server 2 has the maximum waiting time and server 1 has the minimum waiting time. Using the average waiting time, it can be concluded that server 1 is the most effective server among all, since time spent on the queue by the customers is at lowest value.

Computation of Traffic intensity

Traffic intensity (ρ) can be computed using the expression

$$\rho = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(Number\ of\ Servers(S) \times Mean\ Service\ Rate\ (\mu))} = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(3 \times Mean\ Service\ Rate\ (\mu))}$$

To compute traffic intensity for the bank, Tables 31 and 32 were used.

Table 34: Traffic Intensity

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
λ	1.5374 min.	1.3463min.	1.7463min.	1.6437min.	1.3242min.	1.5196min.
S. μ	4.7496min.	5.1795min.	5.4237min.	4.9917min.	5.9814min.	5.2563min.
ρ	0.3237min.	0.2599min.	0.3220min.	0.3293min.	0.2214min.	0.2891min.

The Table 34 analysis above shows that, the intensity was at the peak on the fourth day and the least on the fifth day of the week.

Computation of Probability of Idleness of the system (P_0)

The probability of having exactly zero number of customers in the system or probability that the System is idle is P_0 which is obtained as

$$P_0 = \left[1 + 3\rho + \frac{(3\rho)^2}{2} + \frac{1}{6}(3\rho)^3 + \frac{27\rho^4}{6(1-\rho)}\right]^{-1}$$

For day 1; $\rho = 0.32379$

$$P_0 = \left[1 + 3(0.32369) + \frac{9(0.32369)^2}{2} + \frac{27}{6}(0.32369)^3 + \frac{27(0.32369)^4}{6(1-0.32369)}\right]^{-1}$$

$$P_0 = [1 + 0.9711 + 0.4715 + 0.1526 + 0.0730]^{-1} = 0.3748$$

Similar computation was done for all the points. See Table 4.35.

Table 35: Probability of Idleness of System and Traffic Intensity

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.3237min.	0.2599min.	0.3220min.	0.3293min.	0.2214min.	0.2891min.
P_o	0.3748min.	0.4566min,	0.3768min.	0.3688min.	0.5135min.	0.4173min.

From Table 35, it can be observed that the system has highest possibility of been idle in the fifth day and has the minimum possibility of idleness on the fourth day.

Computation of Probability of having Servers in the System

The probability of having servers in the System is given by

$$P_s = \frac{1}{n!}(s\rho)^n P_0$$

where s is number of servers and n = 3

Table 36: Probability of having Server in the System

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.3237min.	0.2599min.	0.3220min.	0.3293min.	0.2214min.	0.2891min.
P_o	0.3748min.	0.4566min,	0.3768min.	0.3688min.	0.5135min.	0.4173min.
P_3	0.0572min.	0.0361min.	0.0566min.	0.0593min.	0.0251min.	0.0454min.

As shows in Table 36, the probability of having servers in the system is at the peak on the fourth day.

Computation of Probability of servers been busy. P(W(t))

Busy server is necessary in the process as idleness implies redundancy which can be interpreted as wastage. It was computed using 1 minus probability of idleness of the system. Table 4.37 is computed from Table 4.35.

Table 37: Probability of servers been busy

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.3237min.	0.2599min.	0.3220min.	0.3293min.	0.2214min.	0.2891min.
P_o	0.3748min.	0.4566min.	0.3768min.	0.3688min.	0.5135min.	0.4173min.
$(1 - P_o)$	0.6252	0.5434	0.6232	0.6312	0.4865	0.5827

Table 37 shows that in the location servers are busier on the fourth day than any other day and on the average, the servers are busy 58% of the working hours.

Computation of Expected numbers of people waiting to be served E(N)

The expected number of people waiting to be served is given by

$$E(N) = \frac{\rho P_0}{(1-\rho)^2}$$

Table 38: Expected numbers of people waiting to be served

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.3237	0.2599	0.3220	0.3293	0.2214	0.2891
P_o	0.3748	0.4566	0.3768	0.3688	0.5135	0.4173
$(1 - \rho)$	0.6763	0.7401	0.6780	0.6707	0.7786	0.7109
E(N)	0.2653	0.2167	0.2639	0.2699	0.1875	0.2387

From Table 38, on the first day at least 27 per cent of the available customers are expected to wait on queue. 22 percent second day, 26 percent third day, 27 per cent fourth days and 19 per cent fifth days. On the average, for rural area branch of the bank considered, it is expected that 24% of the customers waited on queue to be served.

Computation of Expected time a customer waits for service E(W(t))

The expected duration of waiting time of customer can be computed using;

$$E(W(t)) = \frac{P_s}{\mu S(1-\rho)^2}$$

For day 1, $E(W(t)) = \frac{0.0572}{(1.5832)3(0.4574)^2} = 1.5798$

Table 39: Expected Time a Customer waits for Service

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
P_s	0.0572	0.0361	0.0566	0.0593	0.0251	0.0454
μ	1.5832	1.7265	1.8079	1.6639	1.9938	1.7521
$(1-\rho)^2$	0.4574	0.5477	0.4597	0.4498	0.6062	0.5054
E(w(t)) in min.	0.0263	0.0127	0.0227	0.0264	0.0069	0.0171
E(w(t)) in sec.	1.5798	0.7635	1.3621	1.5847	0.4153	1.0254

As observed in Table 39, customers spent more time waiting for service on the fourth day than any other day as the expected waiting time for the day is approximately 1.5847mins. The least waiting time was observed on the fifth day with waiting time of less than 0.4153mins.

Computation of Conditional Probability of waiting time for service

If a customer has to wait, the expected length of his waiting time

$$E[W(t) \mid W(t) \geq 0] = \frac{1}{\mu s(1-\rho)}$$

Table 40: Conditional Probability of waiting time for service

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
μ	1.5832	1.7265	1.8079	1.6639	1.9938	1.7521
$(1-\rho)$	0.6763	0.7401	0.6780	0.6707	0.7786	0.7109
$E[W(t) \mid W(t) \geq 0]$	0.3113	0.2609	0.2719	0.2987	0.2147	0.2676

Hint: s = 3.

The expected duration of customer waiting for service on the first day if at all there is queue is 0.3113mins and on the average, the customer waiting time is 0.2676.

Computation of Probability that a customer will queue on arrival

This aspect is different from expected waiting time as it shows the possibility of a customer on arrival waiting to be served. The higher the probability of a customer queuing on arrival, the longer the queue in the system. Mathematically, this can be computed using the expression:

Probability that a customer will queue on arrival = $\left(\frac{(\rho s)^s}{s!(1-\rho)}\right) P_0$

For day 1, rho is 0.3237, P_0 is 0.3748 and s=3. Therefore, the probability of customer queuing on arrival is $\left(\frac{(0.3237*3)^3}{3!(1-0.3237)}\right)$ 0.3748 = 0.0846

Similar computation was done for other days. See Table 41.

Table 41: Probability that a customer will queue on arrival

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
ρ	0.3237	0.2599	0.3220	0.3293	0.2214	0.2891
P_o	0.3748	0.4566	0.3768	0.3688	0.5135	0.4173
P(Queuing)	0.0846	0.1067	0.2216	0.2396	0.0627	0.0838

From Table 41, day 4 has the highest probability of customer waiting on arrival before service.

Probability of not queuing on arrival

This can be computed using the expression; 1 minus probability of queuing on arrival. Then, we have;

Table 42: Probability that a customer will not queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P(Queuing)	0.0846	0.1067	0.2216	0.2396	0.0627	0.0838
P(Not Queuing)	0.9154	0.8933	0.7784	0.7604	0.9373	0.9162

Test of goodness of fit using chi-square

This is used to test whether arrival time and service time follow exponential distribution.

Table 43: Mean Arrival Time and Service Time

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Arrival Time	1.5374 min.	1.3463min.	1.7463min.	1.6437min.	1.3242min.	1.5196min.
Service Time	1.5832 min.	1.7265min.	1.8079min.	1.6639min.	1.9938min.	1.7521min.

B. Chi-Square Test of Goodness-of-fit of Arrival Time

To test the hypothesis

H_0: the data follow exponential distribution.

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 44: Observed and Expected for the Goodness-of-fit

Observed [O(x)]	P(x)	Expected [E(x)]	(Obs– Exp.)	$(\mathbf{Obs - Exp})^2$	Chi-Sq. Value
1.5374	0.1840	1.6774	-0.1400	0.0196	0.0117
1.3463	0.2271	2.0703	-0.7240	0.5242	0.2532
1.7463	0.1462	1.3327	0.4136	0.1710	0.1283
1.6437	0.1637	1.4921	0.1516	0.0230	0.0154
1.3242	0.2327	2.1213	-0.797	0.6354	0.2995

From the computation, the Chi-Square calculated is 0.7081.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Comparing both calculated and tabulated values of Chi-Square; 0.7081 and 9.49, there is evidence to accept the null hypothesis and conclude that the data follow exponential distribution.

C. Chi-Square Test of Goodness-of-fit of Service Time

To test the hypothesis
H_0: the data follow exponential distribution.
H_1; the data do not follow exponential distribution.
Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 45: Observed and Expected for the Goodness-of-fit

Observed	P(x)	Expected	Obs–Exp	$(\mathbf{Obs-Exp})^2$	Chi-Sq. Value
1.5832	0.1514	1.5940	-0.0108	0.0001	0.0001
1.7265	0.1276	1.3436	0.3829	0.1466	0.1091
1.8079	0.1158	1.2193	0.5886	0.3464	0.2841
1.6639	0.1375	1.4477	0.2162	0.0467	0.0323
1.9938	0.0928	0.9769	1.0169	1.0341	1.0585

Chi-Square value is 1.4841.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Comparing both calculated and tabulated values of Chi-Square; 1.4841 and 9.49, there is enough evidence to accept the null hypothesis and conclude that the data follow exponential distribution. The queuing model for rural branch of First Bank; Abagana branch, is M/M/3.

Location 4: First Bank (Urban Area)
The bank is also situated at Ziks Avenue, Awka. The bank has 4 servers. See appendix 4 for the data collection

Mean Arrival Rate

$$Mean\ Arrival\ Rate\ (\lambda) = \frac{Total\ Arrival\ Time}{Number\ of\ customers}$$

Table 46: Mean Arrival Rate of Customers in Urban Area of First Bank

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Server 1	3.9913	4.3271	4.5400	5.0110	4.8737	4.5486
Server 2	4.5392	3.9658	4.8247	4.9479	5.1099	4.6775
Server 3	3.9573	4.5258	4.7396	5.0674	3.0215	4.2623
Server 4	3.3333	4.5882	4.7172	4.8854	4.9451	4.4938
Average	3.9553	4.3517	4.7054	4.9779	4.4876	4.4956

In Table 46 above, the mean arrival rate of customers in the first day is 3.95minutes and for the last day of the week is 44.48minutes. On the average, the mean arrival rate is 4.5minutes within the week.

Computation of Mean Service Rate
Mean Service Rate (MSR) can be computed using the expression;

$$Mean\ Service\ Rate\ (\mu) = \frac{Total\ Service\ Time}{Number\ of\ customers}$$

Table 47: Mean Service Rate of Customers in Urban Area of First Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1	4.0357	4.2264	4.5600	5.0106	4.8226	4.5311
Server 2	4.5417	3.9910	4.7895	5.9000	5.0330	4.8510
Server 3	4.0265	4.5670	4.8438	5.0000	5.0323	4.6939
Server 4	3.5667	5.0303	4.8163	4.8660	4.9670	4.6493
Mean	4.0427	4.4537	4.7524	5.1942	4.9637	= 4.6813

Form the result of Table 47, the analysis shows that the mean service rate of customers in the first day is 4minutes and on the average, it is 4.6minutes.

Computation of Mean waiting time

Mean Waiting Time (MWT) can be computed using the expression

$$Mean\ Waiting Time = \frac{Total\ Waiting\ Time}{Number\ of\ customers}$$

Table 48: Mean Waiting Time of Customers in Urban Area of First Bank

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
Server 1	3.6422	1.2150	4.5900	1.2308	0.9634	2.3283
Server 2	2.3684	1.7387	2.5000	1.0206	0.4878	1.6231
Server 3	2.9700	1.0408	4.1547	0.6200	0.6585	1.8888
Server 4	2.4667	1.7000	2.1633	0.8144	1.2073	1.6703
Mean	2.8618	1.4236	3.3520	0.9215	0.8293	1.8776

The analysis of the Table 48 reported that, server 1 has the highest waiting time and server 2 has the least waiting time. Using the waiting time, it can be observed that server 2 is the most efficient server among all since time spent on the queue by the customers is at minimal level.

Computation of Traffic intensity

Traffic intensity (ρ) can be computed using the expression

$$\rho = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(Number\ of\ Servers(S) \times Mean\ Service\ Rate\ (\mu))} = \frac{Mean\ Arrival\ Rate\ (\lambda)}{(4 \times Mean\ Service\ Rate\ (\mu))}$$

To compute traffic intensity for the bank, Tables 4.46 and 4.47 were used.

Table 49: Traffic Intensity

	Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Average**
λ	3.9553	4.3517	4.7054	4.9779	4.4876	4.4956
S. μ	16.1708	17.8148	19.0096	20.7768	19.8548	18.7252
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401

The traffic intensity was at the peak on the third day and the least on the fifty day of the week.

Computation of Probability of Idleness of the system (P_0)
The probability of having exactly zero number of customers in the system or probability that the System is idle is P_0 which is obtained as

$$P_0 = \left[1 + 4\rho + \frac{(4\rho)^2}{2} + \frac{1}{6}(4\rho)^3 + \frac{1}{24}(4\rho)^4 + \frac{256\rho^5}{24(1-\rho)}\right]^{-1}$$

For day 1; $\rho = 0.2446$

$$P_0 = \left[1 + 4(0.2446) + \frac{16(0.2446)^2}{2} + \frac{64}{6}(0.2446)^3 + \frac{256}{24}(0.2446)^4 + \frac{256(0.2446)^5}{24(1-0.2446)}\right]^{-1}$$

$$P_0 = [1 + 0.9784 + 0.4786 + 0.1561 + 0..0382 + 0.0124]^{-1} = 0.3754$$

Similar computation was done for all the points. See Table 4.50.

Table 50: Probability of Idleness of System and Traffic Intensity

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401
P_o	0.3754	0.3759	0.3710	0.3830	0.4046	0.3823

Form table 50 it can be seen that the system has highest possibility of been idle in the fifty day and has the least possibility of idleness on the third day.

Computation of Probability of having Servers in the System
The probability of having servers in the System is given by

$$P_s = \frac{1}{n!}(s\rho)^n P_0$$

where s is number of servers and n = 4

Table 51: Probability of having Server in the System

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401
P_o	0.3754	0.3759	0.3710	0.3830	0.4046	0.3823
P_s	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002

As observed in table 4.6, the probability of having servers in the system is the all the days

Computation of Probability of servers been busy. P(W(t))
Busy server is necessary in the process as idleness implies redundancy which can be interpreted as wastage. It was computed using 1 minus probability of idleness of the system. Table 52 is computed from Table 50.

Table 52: Probability of servers been busy

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401
P_o	0.3754	0.3759	0.3710	0.3830	0.4046	0.3823
$(1 - P_o)$	0.6246	0.6241	0.6290	0.6170	0.5954	0.6177

Table 52 shows that in the location servers are busier on the third day than any other day and on the average, the servers are busy at 62% of the working hours

Computation of Expected numbers of people waiting to be served E(N)
The expected number of people waiting to be served is given by

$$E(N) = \frac{\rho P_0}{(1-\rho)^2}$$

Table 53: Expected numbers of people waiting to be served

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401
P_o	0.3754	0.3759	0.3710	0.3830	0.4046	0.3823
$(1 - \rho)$	0.7554	0.7557	0.7525	0.7604	0.7740	0.7599
E(N)	0.1609	0.1608	0.1622	0.1587	0.1526	0.1590

Form 53, on the first day at least 16 percent of the available customers are expected to wait on queue. 16 percent, 16 percent, 16 percent and 15 percent for second, third, fourth and fifth day respectively. On the average, it is expected that 16 percent of the customers waited on queue to the served.

Computation of Expected time a customer waits for service E(W(t))
The expected duration of waiting time of customer can be computed using equation 11

Table 54: Expected Time a Customer waits for Service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P_s	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
μ	4.0427	4.4537	4.7524	5.1942	4.9637	4.6813
$(1 - \rho)^2$	0.5706	0.5711	0.5663	0.5782	0.5991	0.5774
E(w(t)) in min.	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002
E(w(t)) in sec.	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012

In the Table 54, the analysis reported that customers spent almost the same time on queue for service every day.

Computation of Conditional Probability of waiting time for service
If a customer has to wait, the expected length of his waiting time is given in equation 12. Using the collected data, the result of the computation was as shown in Table 55

Table 55: Conditional Probability of waiting time for service

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
μ	4.0427	4.4537	4.7524	5.1942	4.9637	4.6813
$(1 - \rho)$	0.7554	0.7557	0.7525	0.7604	0.7740	0.7599
$E[W(t) \mid W(t) \geq 0]$	0.0819	0.0743	0.0699	0.0633	0.0651	0.0703

Hint: s = 4.

The expected duration of customer queue for service on the first day if at all there is queue is 0.0819minutes and on the average, the customer waiting time is 0.0703minutes

Computation of Probability that a customer will queue on arrival

This aspect is different from expected waiting time as it shows the possibility of a customer on arrival waiting to be served. Mathematically the probability that a customer will queue on arrival can be computed using the equation 13 and the result of the computation were as shown in table 4.56

Table 56: Probability that a customer will queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
ρ	0.2446	0.2443	0.2475	0.2396	0.2260	0.2401
P_o	0.3754	0.3759	0.3710	0.3830	0.4046	0.3823
P(Queuing)	0.0190	0.0189	0.0197	0.0177	0.0145	0.0178

The Table 56 shows that day 3 has the highest probability of customer waiting on arrival before service.

Probability of not queuing on arrival

This can be computed using the expression; 1 minus probability of queuing on arrival. Then, we have;

Table 57: Probability that a customer will not queue on arrival

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
P(Queuing)	0.0190	0.0189	0.0197	0.0177	0.0145	0.0178
P(Not Queuing)	0.9810	0.9811	0.9803	0.9823	0.9855	0.9822

Test of goodness of fit using Chi-Square

This is used to test whether arrival time and service time follow exponential distribution.

Table 48: Mean Arrival Time and Service Time

	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Arrival Time	3.9553	4.3517	4.7054	4.9779	4.4876	4.4956
Service Time	4.0427	4.4537	4.7524	5.1942	4.9637	4.6813

D. Chi-Square Test of Goodness-of-fit of Arrival Time

To test the hypothesis

H_0: the data follow exponential distribution.

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 59: Observed and Expected for the Goodness-of-fit

Observed [O(x)]	P(x)	Expected [E(x)]	(Obs– Exp.)	$(\mathbf{Obs} - \mathbf{Exp})^2$	Chi-Sq. Value
3.9553	0.0231	0.0152	3.9551	15.6429	38.6153
4.3517	0.0160	0.4322	3.9195	15.3626	35.5461
4.7054	0.0114	0.3089	4.3965	19.3296	62.5846
4.9779	0.0088	0.2384	4.7395	22.4627	94.2161
4.4876	0.0141	0.3798	4.1078	16.8736	44.4224

From the computation, the Chi-Square calculated is 275.38.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Comparing both calculated and tabulated values of Chi-Square; 320.4 and 9.49, there exists enough evidence to reject the null hypothesis and conclude that the data do not follow exponential distribution.

E. Chi-Square Test of Goodness-of-fit of Service Time

To test the hypothesis

H_0: the data follow exponential distribution.

H_1; the data do not follow exponential distribution.

Decision Rule: Accept the null hypothesis if the calculated value of Chi-Square is less than the table value of the test. Otherwise, reject.

Table 60: Observed and Expected for the Goodness-of-fit

Observed	P(x)	Expected	Obs–Exp	$(\mathbf{Obs} - \mathbf{Exp})^2$	Chi-Sq. Value
4.0427	0.000002	0.00006	4.0426	16.3429	248280.06
4.4537	0.0000006	0.00001	4.4537	19.8353	1125738.36
4.7524	0.0000002	0.000006	4.7524	22.5852	3340513.82
5.1942	0.00000005	0.000001	5.1942	26.9797	16455298.12
4.9637	0.0000001	0.000003	4.9637	24.6382	7175785.74

Chi-Square value is 28345616.

Chi-Square tabulated is $\chi^2_{1-\alpha,(k-1)} = \chi^2_{0.95,4} = 9.49$

Comparing both calculated and tabulated values of Chi-Square; 28345616 and 9.49, there is enough evidence to reject the null hypothesis and conclude that the data do not follow Exponential distribution. The queuing model for First Bank urban branch; Zik avenue branch, is G/G/S.

Queuing Model

For each of the locations considered, based on the available data, the most appropriate models are as follows;

Table 61: Queuing Models

	Location	**Queuing Model**
Union Bank	Rural Centre	M/G/S
	Urban Centre	G/G/S
First Bank	Rural Centre	M/M/S
	Urban Centre	G/G/S

The two urban centres have the same model of G/G/S but rural centre of union bank has M/G/S model while rural centre of First bank has M/M/S model. The pattern of the service and banking depend on the location (Rural and Urban) of the branch.

Conclusion

Considering the two banks of interest, First bank has more or higher rate of patronage based on the customers turnover than union bank and irrespective of number of customers, the banks have the same model for urban centre. Therefore, the appropriate model for banks in urban areas is G/G/S. Model for rural centre has the same arrival process but service process differs.

References

Adeleke R.A, Ogunwale O.D. and Halid O.Y (2009): Application of Queuing Theory to Waiting Time of Out ―Patients in Hospital. Pacific Journal of Science and Technology, 10(2)**:**270-274

Bhavin P and Pravin B. (2012) Case Study for Bank ATM Queuing Model. International Journal of Engineering Research and Applications (IJERA) ISSN: 2248-9622. Vol. 2, Issue 5, Pp.1278-1284

Constantinos M. and Mieghem Jan A.V. (2005): Queueing systems with lead time constraints: A fluid-model approach for admission and sequencing control. European Journal of Operational Research 167, Pp 179–207.

Chandra S. and Madhu J. (2013): Special issue on Advances in Queuing Theory and Applications. American Journal of Operational Research. P-ISSN: 2324-6537. e-ISSN: 2324-6545. 3(A).

Disney R. L. (1981): A note on sojourn times in *M/GI/*1 queues with instantaneous Bernoulli feedback. Naval Res. Logist. Quart. 28, 679–684

Emeka E.O. and Favour N. E (2012): Empirical Study of the Use of Automated Teller Machine (ATM) Among Bank Customers In Ibadan Metropolis, South Western Nigeria. European. Journal of Business and. Management, 4(7) pp18-28.

Galit Y. (2014): Erlang-R: A Time-Varying Queue with Reentrant Customers, in Support of Healthcare Staffing. MSOM, Vol.16, No. 2, Pp. 283-299.

Garry W.E (2013): Traffic Engineering with Blocking Probability. http://garywagoner.tripod.com/traffic/BLOCKING.HTML

Harley T. W, Samson O. and Joseph O. I (2014): An Empirical Analysis of Effective Customers Service on Nigeria Banks Profitability. (A Queuing and Regression Approach). Asian Economic and Financial Review. 4(7); 864-876

Hunt, G.C. (*1956)*. "Sequential Arrays of Waiting Lines." Journal of Operations Research, Vol. 4, Pp. 674-683

Jacob M. and Szyszkowski I. (2009): Application of Multi server Queuing to Call Centres. A Thesis Submitted in Partial Fulfillment of the Requirements for Master of

Science in Mathematical Statistics Degree in the Faculty of Science. Rhodes University Faculty of Science

Mahima M., Ranjan B.J and Abhay K. (2014): Application of Queuing in Flow of Information. Special Report. Information Network Lab. Department of Electrical Engineering Indian Institute of Technology Mumbai 400076 India.

Melamed B. (1979): Characterizations of Poisson traffic streams in Jackson queuing networks. Advances in Applied Probability, 11(2):422–438.

Moshe Z. (2016): Introduction to Queuing Theory and Stochastic Teletraffic Model. Electrical Electronics Department, City University of Hong Kong, China.

Nafees N. (2007) Queuing Theory and its Application: Analysis of the sales checkout Operation in ICA Supermarket. M.Sc Thesis, University of Dalarna, retrieved from http://www.statistics.du.se/essays/D07E.Nafees.pdf . 24th October, 2016.

Nomfundo N.D. (2011): Applications of Queuing Theory in Health Care‖, International Journal of Computing and Business Research, vol.2(2), Pp. 231-237.

Ohaneme,C.O., Ohaneme,L.C. Eneh,I.I. and Nwosu, A.W. (2012).Performance Evaluation of Queuing in an Established Petroleum Dispensary System using Simulation Technique. International Journal of Engineering Science and Technology (IJEST), vol.6 (4), Pp. 108-118.

Patel B. and Bhathawala P. (2012): Queuing Model for Banking Industry in Pakistan; International Journal of Engineering Research and Applications (IJERA) Vol. 2, Issue 5, Pp. 123-128.

Paul R.S., Chandrasekher P., Mani V and Thobias S. (2012): Queuing Theory and the Management of Waiting Time in Hospitals: The Case of Anglo Gold Ashanti Hospital in Ghana, International Journal of Academic Research in Business and Social Sciences, vol.4(2), Pp. 75-79.

Perros H.G. and Altiok T. (1986): Approximate analysis of open networks of queues with blocking tandem configurations. IEEE, Transactions on Network Engineering. SE-12, 450-461.

Robert J. B. and Christian T. (2013): Waiting Patiently: An Empirical Study of Queue Abandonment in an Emergency Department. M.Sc Thesis. Department of Biostatistics, Wharton School, University of Pennsylvania, Philadelphia, PA 19104.

Takahashi *et al.* (1980): Study on the deposition of debris flow (2)-Process of formation of debris fan. Annuals, DPRI, 23B-2, pp.443–456 (in Japanese).

Toshiba S, Sanjay K.S and Anil K.K (2013) Application of Queuing Theory for the Improvement of Bank Service. International Journal of Advanced Computational Engineering and Networking, 1(4). Pp. 24-28.