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A MODIFIED RELAXED CQ ALGORITHM FOR THE SOLUTION OF MULTIPLE SPLIT FEASIBILITY PROBLEM IN REAL HILBERT SPACES

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ABSTRACT
Let X_i, Y_i $(i = 1,2,\dots,m)$ be Hilbert spaces, let C_i, K_i $(i = 1,2,\dots,m)$ be closed convex nonempty subsets of X_i and Y_i respectively and let $A_i: X_i \to Y_i$ $(i = 1,2,\dots,m)$ be bounded linear operators. The problem of finding $x_i \epsilon C_i$ such that $A_i x_i \epsilon K_i$ $(i = 1,2,\dots,m)$ is called the multiple split feasibility problem (MSFP). We introduced the multiple split feasibility problem (MSFP) as a natural generalization of the classical split feasibility problem and modify the relaxed CQ algorithm for solution of the split feasibility problem (SFP) for the iterative solution of the MSFP. We establish the weak convergence of this modified algorithm under very mild conditions on the iterative parameters. Using an Ishikawa like process, strong convergence is established.
Keywords: *Hilbert Spaces, Multiple Split, Feasibility Problems, Weak convergence*

Introduction

Let H_1, H_2 be two real Hilbert spaces, $C \subseteq H_1$ and $K \subseteq H_2$ closed convex nonempty subsets and $A : H_1 \to H_2$ a bounded linear operator. The split feasibility problem (SFP) is the problem of finding $x \in C$ such that $Ax \in K$. This was introduced by Censor and Elfving (1994) and was motivated by phase retrieval and other image reconstruction problems in signal recovery and wave theory. The multiple split feasibility problem (MSFP) extends this idea by considering several sets and operators simultaneously. Let X_i, Y_i $(i = 1,2,\dots,m)$ be Hilbert spaces, let C_i, K_i $(i = 1,2,\dots,m)$ be closed convex nonempty subsets of X_i and Y_i respectively, let $A_i: X_i \to Y_i$ $(i = 1,2,\dots,m)$ be bounded linear operators. The problem of finding $x_i \epsilon C_i$ such that $A_i x_i \epsilon K_i$ $(i = 1,2,\dots,m)$ is called the multiple split feasibility problem (MSFP).
Observe that if

$$X = \prod_{i=1}^{m} X_i, \quad Y = \prod_{i=1}^{m} Y_i$$
$$C = \prod_{i=1}^{m} C_i, \quad K = \prod_{i=1}^{m} K_i$$
$$\bar{A} = (A_1, A_1, \dots, A_1)^T, \quad x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$$

then the MSFP can be defined as finding $x^* \in C$ such that $\bar{A}x^* \in K$, using the product topology approach.

(1.1)

The set of solutions of the MSFP, when it exists, is given by

$$\Gamma_i = \{x^* \in C_i \mid A_i x_i^* \epsilon K_i\} = C_i \cap A_i^{-1} K_i, (i = 1,2,\dots,m)$$

or

$$\Gamma = \{x = (x_1^*, x_2^*, \dots, x_m^*) \in \prod_{i=1}^{m} C_i \mid \bar{A}x^* \in \prod_{i=1}^{m} K_i\}$$
$$= (\prod_{i=1}^{m} C_i) \cap (\prod_{i=1}^{m} A_i^{-1} K_i)$$
$$= C \cap \bar{A}^{-1} K$$
$$= \prod_{i=1}^{m} (C \cap \bar{A}^{-1} K)$$
$$= \prod_{i=1}^{m} \Gamma_i$$

The multiple split feasibility problem as we introduced in this paper appears not to have been considered or studied so far in the literature even though it's a natural generalization of the split feasibility problem with real life applications.

Preliminaries

Let X be a metric space. Let $f : D(f) \subseteq X \to R$ and let $\bar{x}_0 \in D(f)$. We say that f is lower semi-continuous at $\bar{x}_0$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$f(\bar{x}_0) - \epsilon < f(x) \quad \forall x \in B(\bar{x}_0, \delta) \cap D(f)$$

is lower semi-continuous if it is lower semi-continuous at every element of its domain.

Equivalently $, f : D(f) \subseteq X \to R$ is lower semicontinuous at $\bar{x}$ if and only if $liminf_{x \to \bar{x}} f(x) \geq f(\bar{x})$.

Let $(X, \|\cdot\| X)$ and $(Y, \|\cdot\| Y)$ be normed spaces and let $T: D(T) \subset X \to Y$ be a linear operator. The operator T is said to be bounded if there is a real constant $c > 0$ such that $\forall x \in D(T) \|Tx\|Y \leq c\|x\|x$.

If T is bounded, the smallest such constant c is called the operator norm of T, denoted by $\| T \|$. If T is not bounded on X, then it is said to be unbounded.

Let H be a Hilbert Space with induced norm $\| \cdot \|$, and let $T : H \to H$ be a mapping, then T is non-expansive if it satisfies the condition

$$\| T(x) - T(y) \| \leq \| x - y \| \ \forall x, y \in H$$

Let H be a Hilbert space. The following identities hold true

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \qquad (2.1)$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \qquad (2.2)$$

for any $x, y \in H$ and $\lambda \in (0, 1)$.

The adjoint operator of A (bounded linear operator), denoted by A^*, is the unique bounded linear operator $A^* : H \to H$ such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \qquad (2.3)$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product in H.

Let K be a nonempty subset of a Hilbert Space H. The distance from a point $x \in H$ to the set K is given by

$$dist(x, K) = inf\{\|x - y\| : y \in K\}$$

A sequence $\{x_n\}$ in a Hilbert space H weakly converges to an element $x \in H$ if, $\forall y \in H$, the following holds

$$\langle x_n, y\rangle \to \langle x, y\rangle \ as\, n \to \infty$$

We denote weak convergence of sequence $\{x_n\}$ to x by $x_n \rightharpoonup x$. The set of weak limits of $\{x_n\}$, denoted by $\omega_w(\{x_n\})$ is defined as

$$\omega_w(\{x_n\}) = \{x \in H \mid \exists \text{ a subsequence } \{x_n\} \text{ of } \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\}$$

In this notation, $x_{n_k} \rightharpoonup x$ denotes weak convergence, meaning $\langle x_n, y\rangle \to \langle x, y\rangle \ \forall y \in H$.

CQ algorithm is an iterative method for solving Split Feasibility Problem(SFP) and was introduced by Byrne (2002). Let x_0 be arbitrary. For $n = 0, 1, \ldots$ and a current estimate x_n, the CQ algorithm is defined as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n \qquad (2.4)$$

where, P_C and P_Q are orthogonal projection operators onto sets C and Q. A^* is the adjoint of A. γ is a positive step-size parameter often chosen based on operator norms.

Algorithm for Multiple Split Feasibility Problem (MSFP)

The relaxed $\boldsymbol{CQ}$ algorithm applied to solve the **SFP** can be modified to make it applicable for solving the **MSFP**. The closed convex sets C_i and Q_i $(i = 1, 2, \ldots, m)$ are defined as level sets.

$$C_i = \{x_i \in X_i : \xi_i(x_i) \leq 0\}$$
$$Q_i = \{y_i \in Y_i : \eta_i(y_i) \leq 0\}$$

where $\xi_i : X_i \to R$ and $\eta_i : Y_i \to R$ are convex functionals. Suppose that ξ_i and η_i are differentiable, that is

$$\partial\xi_i(x_i) = \{z_i \in X_i : \xi_i(u_i) \geq \xi_i(x_i) + \langle u_i - x_i, zi\rangle, \forall\, u_i \in X_i\} \neq 0\ \forall\, x_i \in X_i$$

and

$$\partial\eta_i(y_i = \{wi \in Y_i : \eta_i(v_i) \geq \eta_i(y_i) + \langle v_i - y_i, w_i\rangle, \forall\, v_i \in Y_i\} \neq 0, \forall\, y_i \in Y_i$$

Suppose that C_i, Q_i are bounded sets $\forall i$. This guarantees that if $\{x_{i,n}\} \subset X_i$ and $\{y_{i,n}\} \subset Yi$ are bounded sequences then $\{z_{i,n}\}, \{w_{i,n}\}$ are sequences such that $z_{i,n} \in \partial\xi_i(x_{i,n})$ and $w_{i,n} \in \partial\eta i(y_{i,n})$, then $\{z_{i,n}\}$ and $\{w_{i,n}\}$ are bounded.

Let $\{C_{i,n}\}$and $\{Q_{i,n}\}$ be closed convex sets constructed as follows

$$C_{i,n} = \{z_i \in Xi : \xi_i(x_{i,n}) + \langle \theta_{i,n}, z_i - x_{i,n}\rangle \leq 0\}$$

where $\theta_{i,n} \in \partial\xi_i(x_{i,n})$ and

$$Q_{i,n} = \{w_i \in Y_i : \eta_i(A_i x_{i,n}) + \langle \mu_{i,n}, w_i - A_i x_{i,n}\rangle \leq 0\}$$

where $\mu_{i,n} \in \partial\eta_i(A_i x_{i,n})$.

Define,

$$T_{i,n} = P_{C_{i,n}}(I - \gamma_{i,n}A_i^*(I - P_{Q_{i,n}})A_i) \quad (2.5)$$

and the iteration process $i \in \{1,\ldots,m\}, n \geq 0$

$$x_{i,n+1} = (1 - \alpha)x_{i,n} + \alpha T_{i,n}x_{i,n} \quad (2.6)$$

Gradient Projection Iterative Algorithm

Let H_i $(i = 1,\ldots,m)$ and H be real Hilbert spaces and let C be a closed convex nonempty subset of H. Let $A_i : H_i \to H$ $(i = 1,2,\ldots,m)$ be a bounded linear operators with adjoint A_i^*. The multiple split feasibility problem (MSFP) is the problem of finding $x_i^* \in H_i$ $(i = 1,\ldots,m)$ such that $A_i x_i^* \in C\ \forall\, i = 1,2,\ldots,m$.

Let $\Omega = \{(x_1^*,\ldots,x_m^*) \in \prod_{i=1}^m H_i \mid A_i x_i^* \in C, \forall\, i = 1,2,\ldots,m\}$
$= \{(x_1^*,\ldots,x_m^*) \in \prod_{i=1}^m H_i \mid A_i x_i^* = P_C(A_i x_i^*), \forall\, i = 1,2,\ldots,m\}$ (2.7)

For arbitrary $x_i \in H_i$,

$$x_{i,n+1} = (1 - \alpha)x_{i,n} + \alpha_n y_{i,n},\ n \geq 0\ (i = 1,2,\ldots,m) \quad (2.8)$$
$$y_{i,n} = x_{i,n} - \beta_n A_i^*(A_i x_{i,n} - P_C(A_i x_{i,n})), \quad n \geq 0\ (i = 1,2,\ldots,m)$$
$$(x_1^*,\ldots,x_m^*) \in \Omega$$

Lemma 1. Let K be a non-empty closed convex subsets of real Hilbert space H, then the following are equivalent.

1. $P_K : H \to K$ is the projection operator of $H\ onto\ K$, i.e.

$$\forall\, x \in H\ \| x - P_K x \| \leq \| x - y \|, \quad \forall\, y \in K$$

2. $\forall\, x \in H$, $\langle y - P_K x, x - PKx\rangle \leq 0, \forall\, y \in K$
3. $\forall x \in H$, $\| P_K x - y \|^2 \leq \| x - y \|^2 - \| x - P_K x \|^2$, $\forall\, y \in K$

Lemma 2. Let K be a non-empty closed convex subset of a Hilbert space H.
Let $\{x_n\}$ be a bounded sequence which satisfies the following property.
1. Every weak limit point of $\{x_n\}$ lies in K and
2. $lim_{n\to\infty} \| x_n - x \| = 0\ for\ every\ x \in K$

Then $\{x_n\}$ converges weakly to a point in K

Main Results

Theorem 1. Suppose the MSFP (1.1) is consistent, then the sequence $\{x_n\}$ generated by algorithm (2.5) converges weakly to a solution of the MSFP(1.1).

Proof. If we write

$$T_{i,n} = P_{C_{i,n}}(I - \gamma_{i,n}A_i^*(I - P_{K_{i,n}})A_i)$$

Then $T_{i,n}$ is non-expansive and the relaxed CQ algorithm (2.5) is rewritten as

$$x_{i,n+1} = (1 - \alpha)x_{i,n} + \alpha T_{i,n}x_{i,n}$$
$$= (1 - \alpha)x_{i,n} + \alpha P_{C_{i,n}}\big(I - \gamma_{i,n}A_i^*\big(I - P_{K_{i,n}}\big)A_i\big)x_{i,n}\,, \quad n \geq 0$$

Moreover, we have $T_{i,n}x_i^* = x_i^* \quad \forall\, x_i^* \in \Gamma$. Recall that $x_i^* \in \Gamma$ is the solution set of the MSFP (1.1). It follows that for each $x_i^* \in \Gamma$:

$$\begin{aligned}\|x_{i,n+1} - x_i^*\|^2 &= \|(1-\alpha)x_{i,n} + \alpha P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n} - x_i^*\|^2\\ &= \big\|(1-\alpha)x_{i,n} + \alpha P_{C_{i,n}}\big(I - \gamma_{i,n}A_i^*\,\big(I - P_{K_{i,n}}\big)A_i\big)x_{i,n} - [(1-\alpha)x_{i,n}\\ &\qquad + \alpha x_i^*]\big\|^2\\ &= \|(1-\alpha)(x_{i,n} - x_i^*) + \alpha[P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n} - x_i^*]\|^2\end{aligned}$$

Hence by (2.2)

$$\begin{aligned}\|x_{i,n+1} - x_i^*\|^2 = (1-\alpha)\|x_{i,n} - x_i^*\|^2 &+ \alpha\|P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n} - x_i^*\|^2\\ &- \alpha(1-\alpha)\|x_{i,n} - P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n}\|^2\end{aligned}$$

So,

$$\|x_{i,n+1} - x_i^*\|^2 \leq (1-\alpha)\|x_{i,n} - x_i^*\|^2 + \alpha\|P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n} - x_i^*\|^2 \quad (3.1)$$

Now,

$$\begin{aligned}\|P_{C_{i,n}}(I - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_i)x_{i,n} - x_i^*\|^2 &\leq \|x_{i,n} - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_ix_{i,n} - x_i^*\|^2\\ &= \|(x_{i,n} - x_i^*) - \gamma_{i,n}A_i^*\,(I - P_{K_{i,n}})A_ix_{i,n}\|^2\end{aligned}$$

Hence by (2.1)

$$\begin{aligned}&\|P_{C_{i,n}}(I - \gamma_{i,n}A_i^*)\,(I - P_{K_{i,n}})A_ix_{i,n} - x_i^*\|^2\\ &\leq \|x_{i,n} - x_i^*\|^2 - 2\gamma_{i,n}\langle A_i^*\,(I - P_{K_{i,n}})A_ix_{i,n},\, x_{i,n} - x_i^*\rangle\\ &\qquad + \gamma_{i,n}^2\|A_i^*\|^2\,\|(I - P_{K_{i,n}})A_ix_{i,n}\|^2\\ &= \|x_{i,n} - x_i^*\|^2 - 2\gamma_{i,n}\langle (I - P_{K_{i,n}})A_ix_{i,n}, A_i(x_{i,n} - x_i^*)\rangle\\ &\qquad + \gamma_{i,n}^2\|A_i^*\|\,\|(I - P_{K_{i,n}})A_ix_{i,n}\|^2 \qquad (3.2)\end{aligned}$$

Combining (3.1) and (3.2)

$$\begin{aligned}\|x_{i,n+1} - x_i^*\|^2 &\leq (1-\alpha)\|x_{i,n} - x_i^*\|^2 + \alpha\|x_{i,n} - x_i^*\|^2\\ &\quad -2\alpha\gamma_{i,n}\langle (I - P_{K_{i,n}})A_ix_{i,n}, A_i(x_{i,n} - x_i^*)\rangle + \alpha\gamma_{i,n}^2\|A_i^*\|\,\|(I - P_{K_{i,n}})A_ix_{i,n}\|^2\\ &= \|x_{i,n} - x_i^*\|^2 - 2\alpha\gamma_{i,n}\langle (I - P_{K_{i,n}})A_ix_{i,n}, A_i(x_{i,n} - x_i^*)\rangle\\ &\quad + \alpha\gamma_{i,n}^2\|A_i^*\|\,\|(I - P_{K_{i,n}})A_ix_{i,n}\|^2 \qquad (3.3)\end{aligned}$$

Since $A_ix_i^* \in K \subseteq K_{i,n}$ we have:

$$\langle (I - P_{K_{i,n}})A_ix_{i,n}, A_ix_i^* - P_{K_{i,n}}A_ix_{i,n}\rangle \leq 0 \quad \text{by Lemma 1}$$

This implies that

$$\begin{aligned}\langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_{i,n} - A_ix_i^*\rangle &= \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_{i,n}\rangle - \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_i\rangle\\ &= \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_{i,n}\rangle - \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n}\rangle\\ &\quad + \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n}\rangle - \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_i^*\rangle\end{aligned}$$

$$\begin{aligned}&= \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, A_ix_{i,n} - P_{K_{i,n}}A_ix_{i,n}\rangle + \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n} - A_ix_i^*\rangle\\ &= \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, (1 - P_{K_{i,n}})A_ix_{i,n}\rangle + \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n} - A_ix_i^*\rangle\\ &= \big\|\big(I - P_{K_{i,n}}\big)A_ix_{i,n}\big\|^2 + \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n} - A_ix_i^*\rangle\\ &\Longrightarrow \langle \big(I - P_{K_{i,n}}\big)A_ix_{i,n}, P_{K_{i,n}}A_ix_{i,n} - A_ix_i^*\rangle \geq \big\|\big(I - P_{K_{i,n}}\big)A_ix_{i,n}\big\|^2 \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4) we get

$$\|x_{i,n+1} - x_i^*\|^2 \leq \|x_{i,n} - x_i^*\|^2 - 2\alpha\gamma_{i,n}\|(I - P_{K_{i,n}})A_i x_{i,n}\|^2 + \alpha\gamma_{i,n}^2\|A_i^*\| \, \|(I - P_{K_{i,n}})A_i x_{i,n}\|^2$$

$$= \|x_{i,n} - x_i^*\|^2 - \alpha\gamma_{i,n}(2 - \gamma_{i,n}\|A_i^*\|)\|(I - P_{K_{i,n}})A_i x_{i,n}\|^2 \qquad (3.5)$$

In particular,

$$\lim_{n\to\propto}\|(I - P_{K_{i,n}})A_i x_{i,n}\| = 0 \qquad (3.6)$$

Set $V_{i,n} = A_i^*(I - P_{K_{i,n}})A_i x_{i,n} \to 0$

We next demonstrate that

$$\|x_{i,n+1} - x_{i,n}\| \to 0 \qquad (3.7)$$

To see this, we note that

$$\|x_{i,n+1} - x_{i,n}\|^2 = \|x_{i,n+1} - x_i^* + x_i^* - x_{i,n}\|^2$$

$$= \langle x_{i,n+1} - x_i^* + x_i^* - x_{i,n}, x_{i,n+1} - x_i^* + x_i^* - x_{i,n}\rangle$$

$$= \langle x_{i,n+1} - x_i^*, x_{i,n+1} - x_i^*\rangle + \langle x_{i,n+1} - x_i^*, x_i^* - x_{i,n}\rangle + \langle x_i^* - x_{i,n}, x_{i,n+1} - x_i^*\rangle + \langle x_i^* - x_{i,n}, x_i^* - x_{i,n}\rangle$$

$$= \|x_{i,n+1} - x_i^*\|^2 + 2\langle x_{i,n+1} - x_i^*, x_i^* - x_{i,n+1} - x_{i,n}\rangle + \|x_i^* - x_{i,n}\|^2$$

$$= \|x_{i,n+1} - x_i^*\|^2 + \|x_i^* - x_{i,n}\|^2 + 2[\langle x_{i,n+1} - x_i^*, x_i^* - x_{i,n+1}\rangle + \langle x_{i,n+1} - x_i^*, x_{i,n+1} - x_{i,n}\rangle]$$

$$= \|x_{i,n+1} - x_i^*\|^2 + \|x_i^* - x_{i,n}\|^2 - 2\|x_{i,n+1} - x_i^*\|^2 + 2\langle x_{i,n+1} - x_i^*, x_{i,n+1} - x_{i,n}\rangle$$

$$= \|x_{i,n} - x_i^*\|^2 - \|x_{i,n+1} - x_i^*\|^2 + 2\langle x_{i,n+1} - x_i^*, x_{i,n+1} - x_{i,n}\rangle \qquad (3.8)$$

On the other hand, since

$$x_{i,n+1} = (1 - \alpha)x_{i,n} + \alpha P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n})$$

We have that

$$\langle(1 - \alpha)x_{i,n} + \alpha(\gamma_{i,n} - \gamma_{i,n}V_{i,n}) - x_{i,n+1}, x_i^* - x_{i,n+1}\rangle$$

$$= \left\langle \begin{matrix}(1 - \alpha)x_{i,n} + \alpha(x_{i,n} - \gamma_{i,n}V_{i,n}) - (1 - \alpha)x_{i,n} - \alpha P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n}),\\ x_i^* - (1 - \alpha)x_{i,n} - \alpha P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n})\end{matrix}\right\rangle$$

$$= \left\langle \begin{matrix}\alpha(x_{i,n} - \gamma_{i,n}V_{i,n}) - \alpha P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n}),\\ (1 - \alpha)x_i^* + \alpha x_i^* - (1 - \alpha)x_{i,n} - \alpha P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n})\end{matrix}\right\rangle$$

$$= \alpha\left\langle \begin{matrix}(x_{i,n} - \gamma_{i,n}V_{i,n}) - P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n}),\\ (1 - \alpha)(x_i^* - x_{i,n}) + \alpha x_i^* - P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n})\,]\end{matrix}\right\rangle$$

$$= \alpha^2\left\langle \begin{matrix}(x_{i,n} - \gamma_{i,n}V_{i,n}) - P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n}),\\ \frac{1}{\alpha}[(1 - \alpha)(x_i^* - x_{i,n}) + \alpha x_i^* - P_{C_{i,n}}(x_{i,n} - \gamma_{i,n}V_{i,n})\,]\end{matrix}\right\rangle$$

Let

$$W_n = x_{i,n} - \gamma_{i,n}V_{i,n}, \qquad y_n = \frac{1}{\alpha}[(1 - \alpha)(x_i^* - x_{i,n}) + \alpha x_i^*]$$

Then we have

$$\langle(1 - \alpha)x_{i,n} + \alpha(\gamma_{i,n} - \gamma_{i,n}V_{i,n}) - x_{i,n+1}, x_i^* - x_{i,n+1}\rangle$$

$$= \alpha^2\langle W_n - P_{C_{i,n}}W_n, y_n - P_{C_{i,n}}W_n\rangle \leq 0 \qquad \text{(Lemma 1)}$$

$$\Rightarrow \langle(1 - \alpha)x_{i,n} + \alpha(\gamma_{i,n} - \gamma_{i,n}V_{i,n}) - x_{i,n+1}, x_i^* - x_{i,n+1}\rangle \leq 0$$

$$\Rightarrow \langle(1 - \alpha)x_{i,n} + \alpha\gamma_{i,n} - x_{i,n+1} - \alpha\gamma_{i,n}V_{i,n}, x_i^* - x_{i,n+1}\rangle \leq 0$$

$$\Rightarrow \langle x_{i,n} - x_{i,n+1}\rangle - \langle \alpha\gamma_{i,n}V_{i,n}, x_i^* - x_{i,n+1}\rangle \leq 0$$

$\Longrightarrow \langle x_{i,n} - x_{i,n+1}, x_i^* - x_{i,n+1} \rangle - \langle \alpha\gamma_{i,n} V_{i,n}, x_i^* - x_{i,n+1} \rangle \leq 0$
$\Longrightarrow \langle x_{i,n} - x_{i,n+1}, x_i^* - x_{i,n+1} \rangle \leq \langle \alpha\gamma_{i,n} V_{i,n}, x_i^* - x_{i,n+1} \rangle \leq \alpha\gamma_{i,n} \|V_{i,n}\| \|x_i^* - x_{i,n+1}\| \longrightarrow 0$
We therefore get from (3.8) that

$$\|x_i^* - x_{i,n+1}\| \longrightarrow 0$$

Since $\{x_{i,n}\}$ is bounded which implies that $\{\xi_{i,n}\}$ is bounded, we see that the set of weak limit points of $\{x_{i,n}\}$, $\omega_w(x_{i,n})$ is non-empty. We now show

Claim: $\omega_w(x_{i,n}) \subset \Gamma$
Indeed, assume that $\hat{x}_i \in \omega_w(x_{i,n})$ and $\{x_{i_j,n_j}\}$ is a subsequence of $\{x_{i,n}\}$ which converges weakly to $\hat{x}_i$. Since $x_{i_{j+1},n_{j+1}} \in C_{i_j,n_j}$, we obtain

$$c(x_{i_j,n_j}) + \langle \xi_{i_j,n_j}, x_{i_j,n_{j+1}} - x_{i_j,n_j} \rangle \leq 0$$

Thus,

$$c(x_{i_j,n_j}) \leq -\langle \xi_{i_j,n_j}, x_{i_j,n_{j+1}} - x_{i_j,n_j} \rangle \leq \epsilon \left\|x_{i_j,n_{j+1}} - x_{i_j,n_j}\right\|$$

where $\| \xi_{i,n} \| \leq \xi_i$ $for\ all\ i$.
By virtue of the lower semi-continuity of c, we get by (3.7)

$$c(\hat{x}_i) \leq \liminf_{j\to\infty} c(x_{i_j,n_j}) \leq 0$$

Therefore, $\hat{x}_i \in C$.
Next, we show that $A_i\hat{x}_i \in K_{i,n}$. To see this,
Set $y_{i_j,n_j} = A_i x_{i,n} - P_{K_{i,n}} A_i x_{i,n} \to 0$ and let η be such that $\| y_{i,n} \| \leq \eta$ for each i.

Then since $A_i x_{i_j,n_j} - y_{i_j,n_j} = P_{K_{i_j,n_j}} A_i x_{i_j,n_j} \in K_{i_j,n_j}$ we get

$$k(A_i x_{i_j,n_j}) + \langle \eta_{i_j,n_j}, (A_i x_{i_j,n_j} - y_{i_j,n_j}) - A_i x_{i_j,n_j} \rangle \leq 0$$

Hence,

$$k(A_i x_{i_j,n_j}) \leq \langle \eta_{i_j,n_j}, y_{i_j,n_j} \rangle \leq \eta ||y_{i_j,n_j}|| \to 0$$

By the weak lower semi-continuity of k and the fact that $A_i x_{i_j} \to A_i\hat{x}_i$ weakly, we arrive at the conclusion that

$$k(A_i\hat{x}_i) \leq \liminf_{j\to\infty} k(A_i x_{i_j,n_j}) \leq 0$$

Namely, $A_i\hat{x}_i \in K_i$.
Therefore, $\hat{x}_i \in \Gamma_i$. Now, we can apply **Lemma (2)** to $K_i := \Gamma_i$ to get that the full sequence $\{x_{i,n}\}$ converges weakly to a point in Γ_i.

Theorem 2. Suppose that **MSFP** (1.1) is consistent. Then, the sequence $\{x_{i,n}\}$ of successive approximation generated by algorithm (2.6) converges strongly to a solution of the **MSFP**.

Proof.

$$\begin{aligned}\|y_{i,n} - x_i^*\|^2 &= \|x_{i,n} - x_i^* - \beta_n A_i^* (A_i x_{i,n} - P_{C_i} A_i x_{i,n})\|^2 \\ &= \|x_{i,n} - x_i^*\|^2 - 2\beta_n \langle A_i^* (A_i x_{i,n} - P_{C_i} A_i x_{i,n}), x_{i,n} - x_i^* \rangle \\ &\quad + \beta_n^2 \|A_i^* (A_i x_{i,n} - P_{C_i} A_i x_{i,n})\|^2 \\ &= \|x_{i,n} - x_i^*\|^2 - 2\beta_n \langle A_i^* (A_i x_{i,n} - P_{C_i} A_i x_{i,n}), A_i x_{i,n} - A_i x_i^* \rangle \\ &\quad + \beta_n^2 \|A_i^*\|^2 \|(A_i x_{i,n} - P_{C_i} A_i x_{i,n})\|^2\end{aligned}$$

$$= \|x_{i,n} - x_i^*\|^2 - 2\beta_n\langle A_i^*\left(A_ix_{i,n} - P_{C_i}A_ix_{i,n}\right), A_ix_{i,n} - P_{C_i}A_ix_i^*\rangle$$
$$- 2\beta_n\langle A_i^*\left(A_ix_{i,n} - P_{C_i}A_ix_{i,n}\right), P_{C_i}A_ix_{i,n} - A_ix_i^*\rangle$$
$$+ \beta_n^2\|A_i^*\|^2\|(A_ix_{i,n} - P_{C_i}A_ix_{i,n})\|^2$$
$$= \|x_{i,n} - x_i^*\|^2 - 2\beta_n\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2$$
$$+ \beta_n^2\|A_i^*\|^2\|(A_ix_{i,n} - P_{C_i}A_ix_{i,n})\|^2$$
$$= \|x_{i,n} - x_i^*\|^2 - \beta_n(2 - \beta_n\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2$$

Now,

$$\|x_{i,n+1} - x_i^*\|^2 = \|(1-\alpha_n)x_{i,n} + \alpha_n y_{i,n} - x_i^*\|^2$$
$$= (1-\alpha_n)\|x_{i,n} - x_i^*\|^2 + \alpha_n^2\|y_{i,n} - x_i^*\|^2 - \alpha_n(1-\alpha_n)\|x_{i,n} - y_{i,n}\|^2$$
$$\le (1-\alpha_n)\|x_{i,n} - x_i^*\|^2 + \alpha_n\|y_{i,n} - x_i^*\|^2 - \alpha_n(1-\alpha_n)\|x_{i,n} - y_{i,n}\|^2$$
$$\le (1-\alpha_n)\|x_{i,n} - x_i^*\|^2 + \alpha_n\|y_{i,n} - x_i^*\|^2 - \alpha_n\beta_n(2 - \beta_n\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2$$
$$\le \|x_{i,n} - x_i^*\|^2 - \alpha_n\beta_n(2 - \beta_n\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2$$

Suppose that

$$\alpha_n \in [\alpha, 1], \quad \beta_n \in [\beta, 1] \quad \forall\, n \quad \text{for} \quad \alpha, \beta > 0$$

Then,

$$\|x_{i,n+1} - x_i^*\|^2 \le \|x_{i,n} - x_i^*\|^2 - \alpha\beta(2 - \beta\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2$$

Let,

$$D_n(x_1^*, \dots, x_m^*) = \sum_{i=1}^{m}\|x_{i,n} - x_i^*\|^2$$

Then,

$$D_{n+1}(x_1^*, \dots, x_m^*) = \sum_{i=1}^{m}\|x_{i,n+1} - x_i^*\|^2$$
$$\le \sum_{i=1}^{m}\left(\|x_{i,n} - x_i^*\|^2 - \alpha\beta(2 - \beta\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2\right)$$
$$= D_{n+1}(x_1^*, \dots, x_m^*) - \alpha\beta\sum_{i=1}^{m}\left((2 - \beta\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2\right)$$

So that

$$\alpha\beta\sum_{i=1}^{m}\left((2 - \beta\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2\right) \le D_n(x_1^*, \dots, x_m^*) - D_{n+1}(x_1^*, \dots, x_m^*)$$

$$\alpha\beta\sum_{n\ge 0}\sum_{i=1}^{m}\left((2 - \beta\|A_i^*\|^2)\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2\right)$$
$$\le \sum_{n\ge 0}\{D_n(x_1^*, \dots, x_m^*) - D_{n+1}(x_1^*, \dots, x_m^*)\}$$
$$= D_0(x_1^*, \dots, x_m^*)$$

Thus,

$$\lim_{n\to\infty}\|A_ix_{i,n} - P_{C_i}A_ix_{i,n}\|^2 = 0$$

So that $\{x_{i,n}\}$ is an approximate solution sequence (**ASS**) for the **MSFP**.

$$\lim_{n\to\infty}\|x_{i,n} - y_{i,n}\| = \beta\|A_i^*\| \lim_{n\to\infty}\| A_i x_{i,n} - P_{C_i}A_i x_{i,n}\| = 0$$

Now,

$$D_{n+1}(x_1^*,\ldots,x_m^*) \leq D_n(x_1^*,\ldots,x_m^*), \quad \forall\, n \geq 0$$

Hence $\{D_n(x_1^*,\ldots,x_m^*)\}$ is bounded and $\lim_{n\to\infty} D_n(x_1^*,\ldots,x_m^*)$ exists.

Conclusion

Our modified iterative scheme generalizes some of the existing ones. Our theorem improves, generalizes and extends several known results and our method of proof is of independent interest.

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