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Studies on the Optimum Mechanical Response of Anısotropıc Materials Related to Elastıc Constants *(pp. 203-237)*

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Abstract: In this paper, mechanical and elastic behaviour of anisotropic materials are investigated in order to understand the optimum mechanical behaviour of them in selected directions. For an anisotropic material with known elastic constants, it is possible to choose the best set of elastic constants (effective elastic constants) which determine the optimum mechanical and elastic properties of it. For this reason, bounds on the anisotropic elastic constants have been constructed symbollicaly for all anisotropic elastic symmetries. As illustrative examples, materials from different symmetries are selected and their elastic constants are used to compute bounds on the anisotropic elastic constants. Finally, by examining numerical results of bounds given in tables, it is seen that the materials selected from the same symmetry type which have larger interval between the bounds, are more anisotropic, whereas some materials which have smaller interval between the bounds, are closer to isotropy. The construction of bounds on anisotropic elastic constants is a significant and critical case in design of any engineering and structural materials.

Keywords: anisotropy, bounds, anisotropic elastic constants, anisotropic elastic symmetries.

1 INTRODUCTION

Many materials are anisotropic and inhomogeneous due to the varying composition of their constituents. For instance, polycrystalline materials generally show an elastic anisotropy due to texture and the anisotropy of single crystallites. The polycrystalline and composite materials which show high anisotropy are used in many applications in industry. Anisotropic materials become the material of choice in a variety of engineering applications in the last century and these materials exhibit various symmetry types which can be listed from lower to higher symmetries as triclinic, monoclinic, tetragonal, trigonal, transversely isotropic, cubic. However the type of material symmetry possessed by textured anisotropic materials of geological and biological origin is often a little ambiguous. The ambiguity stems from material variation in the degree of anisotropy in textured materials, from the accuracy of the reported elastic constants and from the accuracy needed for an application. So a person who applies anisotropic elasticity can encounter the following question: For an anisotropic material with known elastic constants, what is the effective elastic constants to represent the material in any specified material symmetry. The answer to that question is presented here.

According to Hill (1952; 1963); the material with triclinic or greater symmetry can be either a polycrystalline material or a composite material or a single crystalline material. In polycrystalline materials, the relation between the elastic properties of single crystals and of `quasi-isotropic' bodies made up of a large number of small single crystallites disposed at all possible orientations has been investigated by Reuss (1929) and Voigt (1928).

Voigt (1928) took the averages of stiffness tensor (c_{ij}) of the crystallites while Reuss (1929) took the averages of compliance tensor (s_{ij}) of the crystallites. Anisotropic Hooke's law is summarized and Kelvin inspired formulation of anisotropic Hooke's law is presented in Dinçkal (2011).

The effective anisotropic elastic constants, k_{eff}, G_{eff}, E_{eff}, ν_{eff} of a triclinic material must satisfy the following bounds:

$$k^R \le k_{eff} \le k^V, \qquad G^R \le G_{eff} \le G^V, \tag{1}$$

$$E^R \le E_{eff} \le E^V, \qquad \nu^V \le \nu_{eff} \le \nu^R. \tag{2}$$

Where the superscript V stands for Voigt bound on k_{eff} , G_{eff} , E_{eff} , ν_{eff} and R stands for Reuss bound on k_{eff} , G_{eff} , E_{eff} , ν_{eff} . The Voigt bound is based on an assumed uniform strain, in other words, Voigt assumes the same deformation in all the grains (uniform deformation) and the Reuss bound is based on an assumed uniform stress which means the same stress in all the grains (Hearmon, 1961; Nye, 1957)

The purpose of this work is to construct bounds on the anisotropic elastic constants for all anisotropic elastic symmetries in terms of elasticity and compliance tensors.

In the present paper, bounds on the anisotropic elastic constants for each material symmetry type have been constructed in section 2. In addition, numerical examples are given in section 3. Finally, in the last section, the results of numerical implementations are discussed and conclusions pertinent to this work are stated.

2 BOUNDS ON THE ANISOTROPIC ELASTIC CONSTANTS

Upper bounds are denoted by k^V(bulk modulus of Voigt) G^V(shear modulus of Voigt) ν^R(Poisson's ratio of Reuss) E^V(Young's modulus of Voigt) Lower bounds are k^R(bulk modulus of Reuss) G^R(shear modulus of Reuss), ν^V(Poisson's ratio of Voigt), E^R(Young's modulus of Reuss). The effective anisotropic elastic constants are k_{eff}(effective bulk modulus), G_{eff}(effective shear modulus), ν_{eff}(effective Poisson's ratio) and E_{eff}(effective Young's modulus). For material symmetry types, the bounds and effective elastic constants are denoted by adding the abbreviation of each symmetry name to the corresponding constants. For instance, for cubic symmetry, upper bounds are denoted by $k^{V,Cub}, G^{V,Cub}$, $\nu^{R,Cub}, E^{V,Cub}$, lower bounds are $k^{R,Cub}, G^{R,Cub}$, $\nu^{V,Cub}, E^{R,Cub}$ and effective anisotropic elastic constants are denoted by k_{eff}^{Cub}, G_{eff}^{Cub}, ν_{eff}^{Cub} and E_{eff}^{Cub}. The effective anisotropic elastic constants, k_{eff}^{Tric}, G_{eff}^{Tric}, ν_{eff}^{Tric} and E_{eff}^{Tric} of a triclinic material must satisfy the following bounds:

$$k^{R,Tric} \le k_{eff}^{Tric} \le k^{V,Tric}, \qquad G^{R,Tric} \le G_{eff}^{Tric} \le G^{V,Tric}, \tag{3}$$

$$E^{R,Tric} \le E_{eff}^{Tric} \le E^{V,Tric}, \qquad \nu^{V,Tric} \le \nu_{eff}^{Tric} \le \nu^{R,Tric}. \tag{4}$$

k_{eff}^{Tric}, G_{eff}^{Tric} , ν_{eff}^{Tric} , E_{eff}^{Tric} are the best sets of elastic constants which should lie between the above bounds. For a triclinic material, the Voigt bounds in terms of elasticity tensors are

$$k^{V,Tric} = \frac{\hat{c}_{11}^{Tric} + \hat{c}_{22}^{Tric} + \hat{c}_{33}^{Tric}}{9} + \frac{2(\hat{c}_{12}^{Tric} + \hat{c}_{23}^{Tric} + \hat{c}_{13}^{Tric})}{9}, \tag{5}$$

$$G^{V,Tric} = \frac{\hat{c}_{11}^{Tric} + \hat{c}_{22}^{Tric} + \hat{c}_{33}^{Tric}}{15} - \frac{\hat{c}_{12}^{Tric} + \hat{c}_{23}^{Tric} + \hat{c}_{13}^{Tric}}{15} + \frac{3(\hat{c}_{44}^{Tric} + \hat{c}_{55}^{Tric} + \hat{c}_{66}^{Tric})}{15}. \tag{6}$$

where $k^{V,Tric}$ represents the bulk modulus of Voigt and $G^{V,Tric}$ represents the shear modulus of Voigt. Young's modulus and Poisson ratio of Voigt are obtained by substituting the values of $k^{V,Tric}$ and $G^{V,Tric}$ in

$$E^{V,Tric} = \frac{27G^V k^V}{3G^V + 9k^V}. \tag{7}$$

$$\nu^{V,Tric} = \frac{1}{2}\left[1 - \frac{3G^V}{3k^V + G^V}\right], \tag{8}$$

where $\nu^{V,Tric}$ and $E^{V,Tric}$ represent the Poisson's ratio and Young's modulus of Voigt respectively and they are also expressed in terms of elasticity tensors as

$$\nu^{V,Tric} = \frac{1}{2}\left[\frac{5(\hat{c}_{11}^{Tric} + \hat{c}_{22}^{Tric} + \hat{c}_{33}^{Tric}) - 8(\hat{c}_{12}^{Tric} + \hat{c}_{23}^{Tric} + \hat{c}_{13}^{Tric}) - 2(\hat{c}_{44}^{Tric} + \hat{c}_{55}^{Tric} + \hat{c}_{66}^{Tric})}{6(\hat{c}_{11}^{Tric} + \hat{c}_{22}^{Tric} + \hat{c}_{33}^{Tric}) - 9(\hat{c}_{12}^{Tric} + \hat{c}_{23}^{Tric} + \hat{c}_{13}^{Tric}) + (\hat{c}_{44}^{Tric} + \hat{c}_{55}^{Tric} + \hat{c}_{66}^{Tric})}\right]. \tag{9}$$

$$E^{V,Tric} = \frac{(i + 2j)(i - j + 3k)}{6i + 9j + 3k}, \tag{10}$$

where $i=\hat{c}_{11}^{Tric}+\hat{c}_{22}^{Tric}+\hat{c}_{33}^{Tric}$, $j=\hat{c}_{12}^{Tric}+\hat{c}_{23}^{Tric}+\hat{c}_{13}^{Tric}$ and $k=\hat{c}_{44}^{Tric}+\hat{c}_{55}^{Tric}+\hat{c}_{66}^{Tric}$.

$k^{V,Tric}$, $G^{V,Tric}$, $\nu^{R,Tric}$, $E^{V,Tric}$ are upper bounds. In similar way, for a triclinic material, Reuss bounds in terms of compliance tensors are

$$k^{R,Tric}=\frac{1}{\hat{s}_{11}^{Tric}+\hat{s}_{22}^{Tric}+\hat{s}_{33}^{Tric}+2(\hat{s}_{12}^{Tric}+\hat{s}_{23}^{Tric}+\hat{s}_{13}^{Tric})}, \tag{11}$$

$$G^{R,Tric}=\frac{15}{4(\hat{s}_{11}^{Tric}+\hat{s}_{22}^{Tric}+\hat{s}_{33}^{Tric})-4(\hat{s}_{12}^{Tric}+\hat{s}_{23}^{Tric}+\hat{s}_{13}^{Tric})+3(\hat{s}_{44}^{Tric}+\hat{s}_{55}^{Tric}+\hat{s}_{66}^{Tric})}. \tag{12}$$

Where $k^{R,Tric}$ represents the bulk modulus of Reuss and $G^{R,Tric}$ represents the shear modulus of Reuss. Poisson ratio and Young's modulus of Reuss are obtained by substituting the values of $k^{R,Tric}$ and $G^{R,Tric}$ in

$$\nu^{R,Tric}=\frac{1}{2}\left[1-\frac{3G^{R}}{3k^{R}+G^{R}}\right], \tag{13}$$

$$E^{R,Tric}=\frac{27G^{R}k^{R}}{3G^{R}+9k^{R}}. \tag{14}$$

Where $\nu^{R,Tric}$ and $E^{R,Tric}$ represent the Poisson's ratio and Young's modulus of Reuss respectively and they are also expressed in terms of compliance tensors as

$$\nu^{R,Tric}=\frac{1}{2}\left[\frac{\hat{s}_{44}^{Tric}+\hat{s}_{55}^{Tric}+\hat{s}_{66}^{Tric}-2(\hat{s}_{11}^{Tric}+\hat{s}_{22}^{Tric}+\hat{s}_{33}^{Tric})-8(\hat{s}_{12}^{Tric}+\hat{s}_{23}^{Tric}+\hat{s}_{13}^{Tric})}{3(\hat{s}_{11}^{Tric}+\hat{s}_{22}^{Tric}+\hat{s}_{33}^{Tric})+2(\hat{s}_{12}^{Tric}+\hat{s}_{23}^{Tric}+\hat{s}_{13}^{Tric})+\hat{s}_{44}^{Tric}+\hat{s}_{55}^{Tric}+\hat{s}_{66}^{Tric}}\right]. \tag{15}$$

$$E^{R,Tric}=\frac{15}{3(\hat{s}_{11}^{Tric}+\hat{s}_{22}^{Tric}+\hat{s}_{33}^{Tric})+2(\hat{s}_{12}^{Tric}+\hat{s}_{23}^{Tric}+\hat{s}_{13}^{Tric})+\hat{s}_{44}^{Tric}+\hat{s}_{55}^{Tric}+\hat{s}_{66}^{Tric}}. \tag{16}$$

$k^{R,Tric}$, $G^{R,Tric}$, $\nu^{V,Tric}$, $E^{R,Tric}$ are lower bounds.

2.1 Bounds for Cubic Symmetry

For cubic symmetry there are three independent Voigt ($\hat{c}_{11}^{V,Cub},\hat{c}_{12}^{V,Cub},\hat{c}_{44}^{V,Cub}$) and Reuss ($\hat{s}_{11}^{R,Cub},\hat{s}_{12}^{R,Cub}$,

$\hat{s}_{44}^{R,Cub}$) elastic constants. These constants are used to construct bounds for cubic symmetry. The upper bounds are denoted by $k^{V,Cub}$, $G^{V,Cub}$, $\nu^{R,Cub}$, $E^{V,Cub}$. The lower bounds are denoted by $k^{R,Cub}$, $G^{R,Cub}$, $\nu^{V,Cub}$, $E^{R,Cub}$. The effective anisotropic elastic constants, k_{eff}^{Cub}, G_{eff}^{Cub}, ν_{eff}^{Cub} and E_{eff}^{Cub} of a cubic material must satisfy the following bounds:

$$k^{R,Cub} \le k_{eff}^{Cub} \le k^{V,Cub}, \qquad G^{R,Cub} \le G_{eff}^{Cub} \le G^{V,Cub}, \tag{17}$$

$$E^{R,Cub} \le E_{eff}^{Cub} \le E^{V,Cub}, \qquad \nu^{V,Cub} \le \nu_{eff}^{Cub} \le \nu^{R,Cub}. \tag{18}$$

$k_{eff}^{Cub}, G_{eff}^{Cub}, \nu_{eff}^{Cub}$, E_{eff}^{Cub} are the best sets of elastic constants of a cubic material which should lie between the above bounds. The bulk modulus of Voigt for cubic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Cub}$, $\hat{c}_{12}^{V,Cub}$ into the equation (5) and it becomes

$$k^{V,Cub} = \frac{\hat{c}_{11}^{V,Cub} + 2\hat{c}_{12}^{V,Cub}}{3}. \tag{19}$$

The shear modulus of Voigt for cubic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Cub}$, $\hat{c}_{12}^{V,Cub}$ $\hat{c}_{44}^{V,Cub}$ into equation (6), then the equation takes the form

$$G^{V,Cub} = \frac{\hat{c}_{11}^{V,Cub} - \hat{c}_{12}^{V,Cub} + 3\hat{c}_{44}^{V,Cub}}{5}. \tag{20}$$

Poisson ratio of Voigt for cubic symmetry is obtained by substituting the values of $k^{V,Cub}$ and $G^{V,Cub}$ in

$$\nu^{V,Cub} = \frac{1}{2}\left[1 - \frac{3G^{V,Cub}}{3k^{V,Cub} + G^{V,Cub}}\right], \tag{21}$$

where $\nu^{V,Cub}$ represents the Poisson's ratio of Voigt for cubic symmetry and it is also expressed by putting the appropriate elastic constants of a cubic material

$$\nu^{V,Cub} = \frac{1}{2}\left[\frac{15\hat{c}_{11}^{V,Cub} - 24\hat{c}_{12}^{V,Cub} - 6\hat{c}_{44}^{V,Cub}}{18\hat{c}_{11}^{V,Cub} - 27\hat{c}_{12}^{V,Cub} + 3\hat{c}_{44}^{V,Cub}}\right]. \tag{22}$$

Young's modulus of Voigt for cubic symmetry is obtained by substituting the values of $k^{V,Cub}$ and $G^{V,Cub}$ in

$$E^{V,Cub} = \frac{27G^{V,Cub}k^{V,Cub}}{3G^{V,Cub}+9k^{V,Cub}}. \tag{23}$$

Where $E^{V,Cub}$ represents the Young's modulus of Voigt for cubic symmetry and it is also expressed by putting the appropriate elastic constants of a cubic material

$$E^{V,Cub} = \left[\frac{(\hat{c}_{11}^{V,Cub}+2\hat{c}_{12}^{V,Cub})(\hat{c}_{11}^{V,Cub}-\hat{c}_{12}^{V,Cub}+3\hat{c}_{44}^{V,Cub})}{6\hat{c}_{11}^{V,Cub}+9\hat{c}_{12}^{V,Cub}+3\hat{c}_{44}^{V,Cub}}\right]. \tag{24}$$

The bulk modulus of Reuss for cubic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Cub}, \hat{s}_{12}^{R,Cub}$ into the equation (11), then it takes the form

$$k^{R,Cub} = \frac{1}{3\hat{s}_{11}^{R,Cub}+6\hat{s}_{12}^{R,Cub}}. \tag{25}$$

The shear modulus of Reuss for cubic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Cub}, \hat{s}_{12}^{R,Cub}, \hat{s}_{44}^{R,Cub}$ into equation (12), then this equation becomes

$$G^{R,Cub} = \frac{15}{12\hat{s}_{11}^{R,Cub}-12\hat{s}_{12}^{R,Cub}+9\hat{s}_{44}^{R,Cub}}. \tag{26}$$

Poisson ratio of Reuss for cubic symmetry is obtained by substituting the values of $k^{R,Cub}$ and $G^{R,Cub}$ in

$$\nu^{R,Cub} = \frac{1}{2}\left[1-\frac{3G^{R,Cub}}{3k^{R,Cub}+G^{R,Cub}}\right]. \tag{27}$$

Where $\nu^{R,Cub}$ represents the Poisson's ratio of Reuss for cubic symmetry and it is also expressed by putting the appropriate elastic constants of a cubic material

$$\nu^{R,Cub} = \frac{1}{2}\left[\frac{3\hat{s}_{44}^{R,Cub} - 24\hat{s}_{12}^{R,Cub} - 6\hat{s}_{11}^{R,Cub}}{3\hat{s}_{44}^{R,Cub} + 9\hat{s}_{11}^{R,Cub} - 6\hat{s}_{12}^{R,Cub}}\right]. \tag{28}$$

Young's modulus of Reuss for cubic symmetry is obtained by substituting the values of $k^{R,Cub}$ and $G^{R,Cub}$ in

$$E^{R,Cub} = \frac{27G^{R,Cub}k^{R,Cub}}{3G^{R,Cub} + 9k^{R,Cub}}. \tag{29}$$

where $E^{R,Cub}$ represents the Young's modulus of Reuss for cubic symmetry and it is also expressed in terms of compliance tensors as

$$E^{R,Cub} = \frac{5}{3\hat{s}_{11}^{R,Cub} + 2\hat{s}_{12}^{R,Cub} + \hat{s}_{44}^{R,Cub}}. \tag{30}$$

2.2 Bounds for Isotropic Symmetry

For isotropic symmetry, there are two independent Voigt ($\hat{c}_{11}^{V,Iso}, \hat{c}_{12}^{V,Iso}$) and Reuss ($\hat{s}_{11}^{R,Iso}, \hat{s}_{12}^{R,Iso}$) elastic constants. These constants are used to construct bounds for isotropic symmetry. The upper bounds are denoted by $k^{V,Iso}$, $G^{V,Iso}$, $\nu^{R,Iso}$, $E^{V,Iso}$. The lower bounds are denoted by $k^{R,Iso}$, $G^{R,Iso}$, $\nu^{V,Iso}$, $E^{R,Iso}$. The effective anisotropic elastic constants, k_{eff}^{Iso}, G_{eff}^{Iso}, ν_{eff}^{Iso} and E_{eff}^{Iso} of an isotropic material must satisfy the following bounds:

$$k^{R,Iso} \le k_{eff}^{Iso} \le k^{V,Iso}, \qquad G^{R,Iso} \le G_{eff}^{Iso} \le G^{V,Iso}, \tag{31}$$

$$E^{R,Iso} \le E_{eff}^{Iso} \le E^{V,Iso}, \qquad \nu^{V,Iso} \le \nu_{eff}^{Iso} \le \nu^{R,Iso}. \tag{32}$$

k_{eff}^{Iso} , G_{eff}^{Iso} , ν_{eff}^{Iso} , E_{eff}^{Iso} are the best sets of elastic constants of a isotropic material which should lie between the above bounds. The bulk modulus of Voigt for isotropic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Iso}$, $\hat{c}_{12}^{V,Iso}$ into equation (5) so the equation takes the form

$$k^{V,Iso} = \frac{\hat{c}_{11}^{V,Iso} + 2\hat{c}_{12}^{V,Iso}}{3}. \qquad (33)$$

The shear modulus of Voigt for isotropic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Iso}$, $\hat{c}_{12}^{V,Iso}$, $\hat{c}_{44}^{V,Iso}$ into equation (6) then it becomes

(where $\hat{c}_{44}^{V,Iso} = \dfrac{(\hat{c}_{11}^{V,Iso} - \hat{c}_{12}^{V,Iso})}{2}$)

$$G^{V,Iso} = \frac{\hat{c}_{11}^{V,Iso} - \hat{c}_{12}^{V,Iso}}{2} = \hat{c}_{44}^{V,Iso}. \qquad (34)$$

Poisson ratio of Voigt for isotropic symmetry is obtained by substituting the values of $k^{V,Iso}$ and $G^{V,Iso}$ in

$$\nu^{V,Iso} = \frac{1}{2}\left[1 - \frac{3G^{V,Iso}}{3k^{V,Iso} + G^{V,Iso}}\right], \qquad (35)$$

where $\nu^{V,Iso}$ represents the Poisson's ratio of Voigt for isotropic symmetry and it is also expressed in terms of elasticity tensors as

$$\nu^{V,Iso} = \frac{1}{2}\left[\frac{12\hat{c}_{11}^{V,Iso} - 21\hat{c}_{12}^{V,Iso}}{13\hat{c}_{11}^{V,Iso} - 19\hat{c}_{12}^{V,Iso}}\right]. \qquad (36)$$

Young's modulus of Voigt for isotropic symmetry is obtained by substituting the values of $k^{V,Iso}$ and $G^{V,Iso}$ in

$$E^{V,Iso} = \frac{27G^{V,Iso}k^{V,Iso}}{3G^{V,Iso} + 9k^{V,Iso}}. \qquad (37)$$

Where $E^{V,Iso}$ represents the Young's modulus of Voigt for isotropic symmetry and it is also expressed in terms of elasticity tensors as

$$E^{V,Iso} = \frac{(\hat{c}_{11}^{V,Iso} + 2\hat{c}_{12}^{V,Iso})(5\hat{c}_{11}^{V,Iso} - 5\hat{c}_{12}^{V,Iso})}{15\hat{c}_{11}^{V,Iso} + 15\hat{c}_{12}^{V,Iso}} . \quad (38)$$

The bulk modulus of Reuss for isotropic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Iso}, \hat{s}_{12}^{R,Iso}$ into the equation (11), it becomes

$$k^{R,Iso} = \frac{1}{3\hat{s}_{11}^{R,Iso} + 6\hat{s}_{12}^{R,Iso}} . \quad (39)$$

The shear modulus of Reuss for isotropic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Iso}, \hat{s}_{12}^{R,Iso}, \hat{s}_{44}^{R,Iso}$ into equation (12), then it takes the form

$$G^{R,Iso} = \frac{1}{2(\hat{s}_{11}^{R,Iso} - \hat{s}_{12}^{R,Iso})} . \quad (40)$$

Poisson ratio of Reuss for isotropic symmetry is obtained by substituting the values of $k^{R,Iso}$ and $G^{R,Iso}$ in

$$\nu^{R,Iso} = \frac{1}{2}\left[1 - \frac{3G^{R,Iso}}{3k^{R,Iso} + G^{R,Iso}}\right] . \quad (41)$$

Where $\nu^{R,Iso}$ represents the Poisson's ratio of Reuss for isotropic symmetry and it is also expressed in terms of compliance tensors as

$$\nu^{R,Iso} = -\frac{\hat{s}_{12}^{R,Iso}}{\hat{s}_{11}^{R,Iso}} . \quad (42)$$

Young's modulus of Reuss for isotropic symmetry is obtained by substituting the values of $k^{R,Iso}$ and $G^{R,Iso}$ in

$$E^{R,Iso} = \frac{27G^{R,Iso}k^{R,Iso}}{3G^{R,Iso} + 9k^{R,Iso}} . \quad (43)$$

Where $E^{R,Iso}$ represents the Young's modulus of Reuss for isotropic symmetry and it is also expressed in terms of compliance tensors

$$E^{R,Iso} = \frac{45}{36\hat{s}_{11}^{R,Iso} + 9\hat{s}_{12}^{R,Iso}}. \quad (44)$$

2.3 **Bounds for** Tetragonal **Symmetry**

For tetragonal symmetry, there are six independent Voigt and Reuss elastic constants. Voigt elastic constants: $\hat{c}_{11}^{V,Tet}, \hat{c}_{12}^{V,Tet}, \hat{c}_{13}^{V,Tet}, \hat{c}_{33}^{V,Tet}, \hat{c}_{44}^{V,Tet}, \hat{c}_{66}^{V,Tet}$ .Reuss elastic constants: $\hat{s}_{11}^{R,Tet}, \hat{s}_{12}^{R,Tet}, \hat{s}_{13}^{R,Tet}, \hat{s}_{33}^{R,Tet}, \hat{s}_{44}^{R,Tet}, \hat{s}_{66}^{R,Tet}$ .These constants are used to construct bounds for tetragonal symmetry. The upper bounds are denoted by $k^{V,Tet}$, $G^{V,Tet}$, $\nu^{R,Tet}$, $E^{V,Tet}$. The lower bounds are denoted by $k^{R,Tet}$, $G^{R,Tet}$, $\nu^{V,Tet}$, $E^{R,Tet}$. The effective anisotropic elastic constants, k_{eff}^{Tet}, G_{eff}^{Tet}, ν_{eff}^{Tet} and E_{eff}^{Tet} of a tetragonal material must satisfy the following bounds:

$$k^{R,Tet} \le k_{eff}^{Tet} \le k^{V,Tet}, \qquad G^{R,Tet} \le G_{eff}^{Tet} \le G^{V,Tet}, \quad (45)$$

$$E^{R,Tet} \le E_{eff}^{Tet} \le E^{V,Tet}, \qquad \nu^{V,Tet} \le \nu_{eff}^{Tet} \le \nu^{R,Tet}. \quad (46)$$

k_{eff}^{Tet} , G_{eff}^{Tet} , ν_{eff}^{Tet} and E_{eff}^{Tet} are the best sets of elastic constants of a tetragonal material which should lie between the above bounds.

The bulk modulus of Voigt for tetragonal symmetry can be obtained by substituting $\hat{c}_{11}^{V,Tet}$, $\hat{c}_{12}^{V,Tet}$, $\hat{c}_{13}^{V,Tet}$, $\hat{c}_{33}^{V,Tet}$ into the equation (5), then it takes the form

$$k^{V,Tet} = \frac{2\hat{c}_{11}^{V,Tet} + \hat{c}_{33}^{V,Tet}}{9} + \frac{2(\hat{c}_{12}^{V,Tet} + 2\hat{c}_{13}^{V,Tet})}{9}. \quad (47)$$

The shear modulus of Voigt for tetragonal symmetry can be obtained by substituting $\hat{c}_{11}^{V,Tet}$, $\hat{c}_{12}^{V,Tet}, \hat{c}_{13}^{V,Tet}, \hat{c}_{33}^{V,Tet}$, $\hat{c}_{44}^{V,Tet}, \hat{c}_{66}^{V,Te}$ into equation (6), then this equation becomes

$$G^{V,Tet} = \frac{2\hat{c}_{11}^{V,Tet} + \hat{c}_{33}^{V,Tet}}{15} - \frac{\hat{c}_{12}^{V,Tet} + 2\hat{c}_{13}^{V,Tet}}{15} + \frac{3(2\hat{c}_{44}^{V,Tet} + \hat{c}_{66}^{V,Tet})}{15}. \quad (48)$$

Poisson ratio of Voigt for tetragonal symmetry is obtained by substituting the values of $k^{V,Tet}$ and $G^{V,Tet}$ in

$$\nu^{V,Tet} = \frac{1}{2}\left[1 - \frac{3G^{V,Tet}}{3k^{V,Tet} + G^{V,Tet}}\right], \tag{49}$$

where $\nu^{V,Tet}$ represents the Poisson's ratio of Voigt for tetragonal symmetry and it is also expressed in terms of elasticity tensors as

$$\nu^{V,Tet} = \frac{1}{2}\left[\frac{10\hat{c}_{11}^{V,Tet} + 5\hat{c}_{33}^{V,Tet} - 8\hat{c}_{12}^{V,Tet} - 16\hat{c}_{13}^{V,Tet} - 4\hat{c}_{44}^{V,Tet} - 2\hat{c}_{66}^{V,Tet}}{12\hat{c}_{11}^{V,Tet} + 6\hat{c}_{33}^{V,Tet} - 9\hat{c}_{12}^{V,Tet} - 18\hat{c}_{13}^{V,Tet} + 2\hat{c}_{44}^{V,Tet} + \hat{c}_{66}^{V,Tet}}\right]. \tag{50}$$

Young's modulus of Voigt for tetragonal symmetry is obtained by substituting the values of $k^{V,Tet}$ and $G^{V,Tet}$ in

$$E^{V,Tet} = \frac{27G^{V,Tet}k^{V,Tet}}{3G^{V,Tet} + 9k^{V,Tet}}, \tag{51}$$

where $E^{V,Tet}$ represents the Young's modulus of Voigt for tetragonal symmetry and in terms of elasticity tensors, it is also expressed as

$$E^{V,Tet} = \frac{(l + 2\hat{c}_{12}^{V,Tet} + 4\hat{c}_{13}^{V,Tet})(l - \hat{c}_{12}^{V,Tet} - 2\hat{c}_{13}^{V,Tet} + 6\hat{c}_{44}^{V,Tet} + 3\hat{c}_{66}^{V,Tet})}{6l + 9\hat{c}_{12}^{V,Tet} + 18\hat{c}_{13}^{V,Tet} + 6\hat{c}_{44}^{V,Tet} + 3\hat{c}_{66}^{V,Tet}}, \tag{52}$$

where $l = 2\hat{c}_{11}^{V,Tet} + \hat{c}_{33}^{V,Tet}$. The bulk modulus of Reuss for tetragonal symmetry can be obtained by substituting $\hat{s}_{11}^{R,Tet}, \hat{s}_{12}^{R,Tet}, \hat{s}_{13}^{R,Tet}, \hat{s}_{33}^{R,Tet}$ into the equation (11), then it takes the form

$$k^{R,Tet} = \frac{1}{2\hat{s}_{11}^{R,Tet} + \hat{s}_{33}^{R,Tet} + 2(\hat{s}_{12}^{R,Tet} + 2\hat{s}_{13}^{R,Tet})}. \tag{53}$$

The shear modulus of Reuss for tetragonal symmetry can be obtained by substituting $\hat{s}_{11}^{R,Tet}, \hat{s}_{12}^{R,Tet}, \hat{s}_{13}^{R,Tet}, \hat{s}_{33}^{R,Tet}, \hat{s}_{44}^{R,T\ e\ t}, \hat{s}_{66}^{R,T\ e}$ into equation (12), then this equation becomes

$$G^{R,Tet} = \frac{15}{4(2\hat{s}_{11}^{R,Tet} + \hat{s}_{33}^{R,Tet}) - 4(\hat{s}_{12}^{R,Tet} + 2\hat{s}_{13}^{R,Tet}) + 3(2\hat{s}_{44}^{R,Tet} + \hat{s}_{66}^{R,Tet})}. \tag{54}$$

Poisson ratio of Reuss for tetragonal symmetry is obtained by substituting the values of

$k^{R,Tet}$ and $G^{R,Tet}$ in

$$\nu^{R,Tet} = \frac{1}{2}\left[1 - \frac{3G^{R,Tet}}{3k^{R,Tet} + G^{R,Tet}}\right]. \tag{55}$$

Where $\nu^{R,Tet}$ represents the Poisson's ratio of Reuss for tetragonal symmetry and it is also expressed by putting the appropriate elastic constants of a tetragonal material

$$\nu^{R,Tet} = \frac{1}{2}\left[\frac{2\hat{s}_{44}^{R,Tet} + \hat{s}_{66}^{R,Tet} - 4\hat{s}_{11}^{R,Tet} - 2\hat{s}_{33}^{R,Tet} - 8\hat{s}_{12}^{R,Tet} - 16\hat{s}_{13}^{R,Tet}}{6\hat{s}_{11}^{R,Tet} + 3\hat{s}_{33}^{R,Tet} + 2\hat{s}_{12}^{R,Tet} + 4\hat{s}_{13}^{R,Tet} + 2\hat{s}_{44}^{R,Tet} + \hat{s}_{66}^{R,Tet}}\right]. \tag{56}$$

Young's modulus of Reuss for tetragonal symmetry is obtained by substituting the values of $k^{R,Tet}$ and $G^{R,Tet}$ in

$$E^{R,Tet} = \frac{27G^{R,Tet}k^{R,Tet}}{3G^{R,Tet} + 9k^{R,Tet}}. \tag{57}$$

Where $E^{R,Tet}$ represents the Young's modulus of Reuss for tetragonal symmetry and it is also expressed by putting the appropriate elastic constants of a tetragonal material

$$E^{R,Tet} = \frac{45}{(18\hat{s}_{11}^{R,Tet} + 9\hat{s}_{33}^{R,Tet} + 6\hat{s}_{12}^{R,Tet} + 12\hat{s}_{13}^{R,Tet} + 6\hat{s}_{44}^{R,Tet} + 3\hat{s}_{66}^{R,Tet})}. \tag{58}$$

2.4 Bounds for Transversely Isotropic Symmetry

For transversely isotropic symmetry, there are five independent Voigt and Reuss elastic constants Voigt elastic constants: $\hat{c}_{11}^{V,Trans}, \hat{c}_{12}^{V,Trans}, \hat{c}_{13}^{V,Trans}, \hat{c}_{33}^{V,Trans}, \hat{c}_{44}^{V,Trans}$. Reuss elastic constants: $\hat{s}_{11}^{R,Trans}$, $\hat{s}_{12}^{R,Trans}, \hat{s}_{13}^{R,Trans}, \hat{s}_{33}^{R,Trans}, \hat{s}_{44}^{R,Trans}$. These constants are used to construct bounds for transversely isotropic symmetry. The upper bounds are denoted by $k^{V,Trans}$, $G^{V,Trans}$, $\nu^{R,Trans}$, $E^{V,Trans}$. The lower bounds are denoted by $k^{R,Trans}$, $G^{R,Trans}$, $\nu^{V,Trans}$, $E^{R,Trans}$. The effective anisotropic elastic constants, k_{eff}^{Trans}, G_{eff}^{Trans}, ν_{eff}^{Trans} and E_{eff}^{Trans} of a transversely isotropic material must satisfy the following bounds:

$$k^{R,Trans} \le k_{eff}^{Trans} \le k^{V,Trans}, \qquad G^{R,Trans} \le G_{eff}^{Trans} \le G^{V,Trans}, \tag{59}$$

$$E^{R,Trans} \le E_{eff}^{Trans} \le E^{V,Trans}, \qquad \nu^{V,Trans} \le \nu_{eff}^{Trans} \le \nu^{R,Trans}. \qquad (60)$$

k_{eff}^{Trans} , G_{eff}^{Trans} , ν_{eff}^{Trans} , E_{eff}^{Trans} are the best sets of elastic constants of a transversely isotropic material which should lie between the above bounds. The bulk modulus of Voigt for transversely isotropic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Trans}$, $\hat{c}_{12}^{V,Trans}$, $\hat{c}_{13}^{V,Trans}$, $\hat{c}_{33}^{V,Trans}$ into the equation (5), then it takes the form

$$k^{V,Trans} = \frac{2\hat{c}_{11}^{V,Trans} + \hat{c}_{33}^{V,Trans}}{9} + \frac{2(\hat{c}_{12}^{V,Trans} + 2\hat{c}_{13}^{V,Trans})}{9}. \qquad (61)$$

The shear modulus of Voigt for transversely isotropic symmetry can be obtained by substituting $\hat{c}_{11}^{V,Trans}$, $\hat{c}_{12}^{V,Trans}$, $\hat{c}_{13}^{V,Trans}$, $\hat{c}_{33}^{V,Trans}$, $\hat{c}_{44}^{V,Trans}$ into equation (6) then this equation becomes

$$G^{V,Trans} = \frac{7\hat{c}_{11}^{V,Trans} - 5\hat{c}_{12}^{V,Trans}}{30} + \frac{\hat{c}_{33}^{V,Trans} - 2\hat{c}_{13}^{V,Trans}}{15} + \frac{2\hat{c}_{44}^{V,Trans}}{5}. \qquad (62)$$

Poisson ratio of Voigt for transversely isotropic symmetry is obtained by substituting the values of $k^{V,Trans}$ and $G^{V,Trans}$ in

$$\nu^{V,Trans} = \frac{1}{2}\left[1 - \frac{3G^{V,Trans}}{3k^{V,Trans} + G^{V,Trans}}\right]. \qquad (63)$$

Where $\nu^{V,Trans}$ represents the Poisson's ratio of Voigt for transversely isotropic symmetry and it is also expressed by substituting the appropriate elastic constants of a transversely isotropic material

$$\nu^{V,Trans} = \frac{1}{2}\left[\frac{9\hat{c}_{11}^{V,Trans} + 5\hat{c}_{33}^{V,Trans} - 7\hat{c}_{12}^{V,Trans} - 16\hat{c}_{13}^{V,Trans} - 4\hat{c}_{44}^{V,Trans}}{\frac{25}{2}\hat{c}_{11}^{V,Trans} + 6\hat{c}_{33}^{V,Trans} - \frac{19}{2}\hat{c}_{12}^{V,Trans} - 18\hat{c}_{13}^{V,Trans} + 2\hat{c}_{44}^{V,Trans}}\right]. \qquad (64)$$

Young's modulus of Voigt for transversely isotropic symmetry is obtained by substituting the values of $k^{V,Trans}$ and $G^{V,Trans}$ in

$$E^{V,Trans}=\frac{27G^{V,Trans}k^{V,Trans}}{3G^{V,Trans}+9k^{V,Trans}}. \qquad (65)$$

Where $E^{V,Trans}$ represents the Young's modulus of Voigt for transversely isotropic symmetry and it is also expressed by putting the appropriate elastic constants of a transversely isotropic material

$$E^{V,Trans}=\frac{mn}{o}, \qquad (66)$$

where $m=2\hat{c}_{11}^{V,Trans}+\hat{c}_{33}^{V,Trans}+2\hat{c}_{12}^{V,Trans}+4\hat{c}_{13}^{V,Trans}$,

$n=\frac{7}{2}\hat{c}_{11}^{V,Trans}-\frac{5}{2}\hat{c}_{12}^{V,Trans}+\hat{c}_{33}^{V,Trans}-2\hat{c}_{13}^{V,Trans}+6\hat{c}_{44}^{V,Trans}$ and

$o=\frac{27}{2}\hat{c}_{11}^{V,Trans}+6\hat{c}_{33}^{V,Trans}+\frac{15}{2}\hat{c}_{12}^{V,Trans}+18\hat{c}_{13}^{V,Trans}+6\hat{c}_{44}^{V,Trans}$.

The bulk modulus of Reuss for transversely isotropic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Trans},\hat{s}_{12}^{R,Trans},\hat{s}_{13}^{R,Trans},\hat{s}_{33}^{R,Trans}$ into the equation (11), then it takes the form

$$k^{R,Trans}=\frac{1}{2\hat{s}_{11}^{R,Trans}+\hat{s}_{33}^{R,Trans}+2(\hat{s}_{12}^{R,Trans}+2\hat{s}_{13}^{R,Trans})}. \qquad (67)$$

The shear modulus of Reuss for transversely isotropic symmetry can be obtained by substituting $\hat{s}_{11}^{R,Trans},\hat{s}_{12}^{R,Trans},\hat{s}_{13}^{R,Trans},\hat{s}_{33}^{R,Trans}$, $\hat{s}_{44}^{R,Trans}$ into equation (12), then equation (12) takes the form

$$G^{R,Trans}=\frac{15}{14\hat{s}_{11}^{R,Trans}+4\hat{s}_{33}^{R,Trans}-10\hat{s}_{12}^{R,Trans}-8\hat{s}_{13}^{R,Trans}+6\hat{s}_{44}^{R,Trans}}. \qquad (68)$$

Poisson ratio of Reuss for transversely isotropic symmetry is obtained by substituting the values of $k^{R,Trans}$ and $G^{R,Trans}$ in

$$\nu^{R,Trans}=\frac{1}{2}\left[1-\frac{3G^{R,Trans}}{3k^{R,Trans}+G^{R,Trans}}\right], \qquad (69)$$

where $\nu^{R,Trans}$ represents the Poisson's ratio of Reuss for transversely isotropic symmetry and it is also expressed in terms of the appropriate elastic constants

$$\nu^{R,Trans} = \frac{1}{2}\left[\frac{2\hat{s}_{44}^{R,Trans} - 2\hat{s}_{33}^{R,Trans} - 10\hat{s}_{12}^{R,Trans} - 16\hat{s}_{13}^{R,Trans}}{8\hat{s}_{11}^{R,Trans} + 3\hat{s}_{33}^{R,Trans} + 4\hat{s}_{13}^{R,Trans} + 2\hat{s}_{44}^{R,Trans}}\right]. \quad (70)$$

Young's modulus of Reuss for transversely isotropic symmetry is obtained by substituting the values of $k^{R,Trans}$ and $G^{R,Trans}$ in

$$E^{R,Trans} = \frac{27G^{R,Trans}k^{R,Trans}}{3G^{R,Trans} + 9k^{R,Trans}}. \quad (71)$$

Where $E^{R,Trans}$ represents the Young's modulus of Reuss for transversely isotropic symmetry and it is also expressed in terms of the appropriate elastic constants of a transversely isotropic material

$$E^{R,Trans} = \frac{45}{24\hat{s}_{11}^{R,Trans} + 9\hat{s}_{33}^{R,Trans} + 12\hat{s}_{13}^{R,Trans} + 6\hat{s}_{44}^{R,Trans}}. \quad (72)$$

2.5 Bounds for Trigonal Symmetry

For trigonal symmetry, there are six independent Voigt and Reuss elastic constants. Voigt elastic constants: $\hat{c}_{11}^{V,Trig}, \hat{c}_{12}^{V,Trig}, \hat{c}_{13}^{V,Trig}, \hat{c}_{14}^{V,Trig}, \hat{c}_{33}^{V,Trig}, \hat{c}_{44}^{V,Trig}$. Reuss elastic constants: $\hat{s}_{11}^{R,Trig}, \hat{s}_{12}^{R,Trig}, \hat{s}_{13}^{R,Trig}, \hat{s}_{14}^{R,Trig}, \hat{s}_{33}^{R,Trig}, \hat{s}_{44}^{R,Trig}$. These constants are used to construct bounds for trigonal symmetry. The upper bounds are denoted by $k^{V,Trig}$, $G^{V,Trig}$, $\nu^{R,Trig}$, $E^{V,Trig}$. The lower bounds are denoted by $k^{R,Trig}$, $G^{R,Trig}$, $\nu^{V,Trig}$, $E^{R,Trig}$. The effective anisotropic elastic constants, k_{eff}^{Trig}, G_{eff}^{Trig}, ν_{eff}^{Trig} and E_{eff}^{Trig} of a trigonal material must satisfy the following bounds:

$$k^{R,Trig} \le k_{eff}^{Trig} \le k^{V,Trig}, \qquad G^{R,Trig} \le G_{eff}^{Trig} \le G^{V,Trig}, \quad (73)$$

$$E^{R,Trig} \le E_{eff}^{Trig} \le E^{V,Trig}, \qquad \nu^{V,Trig} \le \nu_{eff}^{Trig} \le \nu^{R,Trig}. \quad (74)$$

k_{eff}^{Trig} , G_{eff}^{Trig} , ν_{eff}^{Trig} , E_{eff}^{Trig} are the best sets of elastic constants of a trigonal material which should lie between the above bounds. The bulk modulus of Voigt for trigonal symmetry can be obtained by substituting $\hat{c}_{11}^{V,Trig}$, $\hat{c}_{12}^{V,Trig}$, $\hat{c}_{13}^{V,Trig}$, $\hat{c}_{33}^{V,Trig}$ into the equation (5), then it takes the form

$$k^{V,Trig} = \frac{2\hat{c}_{11}^{V,Trig} + \hat{c}_{33}^{V,Trig}}{9} + \frac{2(\hat{c}_{12}^{V,Trig} + 2\hat{c}_{13}^{V,Trig})}{9}. \quad (75)$$

The shear modulus of Voigt for trigonal symmetry can be obtained by substituting $\hat{c}_{11}^{V,Trig}$, $\hat{c}_{12}^{V,Trig}, \hat{c}_{13}^{V,Trig}, \hat{c}_{33}^{V,Trig}$, $\hat{c}_{44}^{V,Trig}$ into equation (6) so it becomes

$$G^{V,Trig} = \frac{7\hat{c}_{11}^{V,Trig} - 5\hat{c}_{12}^{V,Trig}}{30} + \frac{\hat{c}_{33}^{V,Trig} - 2\hat{c}_{13}^{V,Trig}}{15} + \frac{2\hat{c}_{44}^{V,Trig}}{5}. \quad (76)$$

Poisson ratio of Voigt for trigonal symmetry is obtained by substituting the values of $k^{V,Trig}$ and $G^{V,Trig}$ in

$$\nu^{V,Trig} = \frac{1}{2}\left[1 - \frac{3G^{V,Trig}}{3k^{V,Trig} + G^{V,Trig}}\right]. \quad (77)$$

Where $\nu^{V,Trig}$ represents the Poisson's ratio of Voigt for trigonal symmetry and it is also expressed by putting the appropriate elastic constants of a trigonal material

$$\nu^{V,Trig} = \frac{1}{2}\left[\frac{9\hat{c}_{11}^{V,Trig} + 5\hat{c}_{33}^{V,Trig} - 7\hat{c}_{12}^{V,Trig} - 16\hat{c}_{13}^{V,Trig} - 4\hat{c}_{44}^{V,Trig}}{\frac{25}{2}\hat{c}_{11}^{V,Trig} + 6\hat{c}_{33}^{V,Trig} - \frac{19}{2}\hat{c}_{12}^{V,Trig} - 18\hat{c}_{13}^{V,Trig} + 2\hat{c}_{44}^{V,Trig}}\right]. \quad (78)$$

Young's modulus of Voigt for trigonal symmetry is obtained by substituting the values of $k^{V,Trig}$ and $G^{V,Trig}$ in

$$E^{V,Trig} = \frac{27G^{V,Trig}k^{V,Trig}}{3G^{V,Trig} + 9k^{V,Trig}}. \quad (79)$$

Where $E^{V,Trig}$ represents the Young's modulus of Voigt for trigonal symmetry and it is also expressed in terms of the appropriate elastic constants of a trigonal material

$$E^{V,Trig} = \frac{pr}{s}, \quad (80)$$

where $p = 2\hat{c}_{11}^{V,Trig} + \hat{c}_{33}^{V,Trig} + 2\hat{c}_{12}^{V,Trig} + 4\hat{c}_{13}^{V,Trig}$,

$$r = \frac{7}{2}\hat{c}_{11}^{V,Trig} - \frac{5}{2}\hat{c}_{12}^{V,Trig} + \hat{c}_{33}^{V,Trig} - 2\hat{c}_{13}^{V,Trig} + 6\hat{c}_{44}^{V,Trig}$$

and $s=\frac{27}{2}\hat{c}_{11}^{V,Trig}+6\hat{c}_{33}^{V,Trig}+\frac{15}{2}\hat{c}_{12}^{V,Trig}+18\hat{c}_{13}^{V,Trig}+6\hat{c}_{44}^{V,Trig}$.

The bulk modulus of Reuss for trigonal symmetry can be obtained by substituting $\hat{s}_{11}^{R,Trig}, \hat{s}_{12}^{R,Trig}, \hat{s}_{13}^{R,Trig}, \hat{s}_{33}^{R,Trig}$ into the equation (11), and then it takes the form

$$k^{R,Trig}=\frac{1}{2\hat{s}_{11}^{R,Trig}+\hat{s}_{33}^{R,Trig}+2(\hat{s}_{12}^{R,Trig}+2\hat{s}_{13}^{R,Trig})}. \qquad (81)$$

The shear modulus of Reuss for trigonal symmetry can be obtained by substituting $\hat{s}_{11}^{R,Trig}, \hat{s}_{12}^{R,Trig}, \hat{s}_{13}^{R,Trig}, \hat{s}_{33}^{R,Trig}, \hat{s}_{44}^{R,Trig}$ into equation (12), then it becomes

$$G^{R,Trig}=\frac{15}{14\hat{s}_{11}^{R,Trig}+4\hat{s}_{33}^{R,Trig}-10\hat{s}_{12}^{R,Trig}-8\hat{s}_{13}^{R,Trig}+6\hat{s}_{44}^{R,Trig}}. \qquad (82)$$

Poisson ratio of Reuss for trigonal symmetry is obtained by substituting the values of $k^{R,Trig}$ and $G^{R,Trig}$ in

$$\nu^{R,Trig}=\frac{1}{2}\left[1-\frac{3G^{R,Trig}}{3k^{R,Trig}+G^{R,Trig}}\right]. \qquad (83)$$

Where $\nu^{R,Trig}$ represents the Poisson's ratio of Reuss for trigonal symmetry and it is also expressed in terms of the appropriate elastic constants of a trigonal material

$$\nu^{R,Trig}=\frac{1}{2}\left[\frac{2\hat{s}_{44}^{R,Trig}-2\hat{s}_{33}^{R,Trig}-10\hat{s}_{12}^{R,Trig}-16\hat{s}_{13}^{R,Trig}}{8\hat{s}_{11}^{R,Trig}+3\hat{s}_{33}^{R,Trig}+4\hat{s}_{13}^{R,Trig}+2\hat{s}_{44}^{R,Trig}}\right]. \qquad (84)$$

Young's modulus of Reuss for trigonal symmetry is obtained by substituting the values of $k^{R,Trig}$ and $G^{R,Trig}$ in

$$E^{R,Trig}=\frac{27G^{R,Trig}k^{R,Trig}}{3G^{R,Trig}+9k^{R,Trig}}. \qquad (85)$$

Where $E^{R,Trig}$ represents the Young's modulus of Reuss for trigonal symmetry and it is also expressed in terms of compliance tensors as

$$E^{R,Trig} = \frac{45}{24\hat{s}_{11}^{R,Trig} + 9\hat{s}_{33}^{R,Trig} + 12\hat{s}_{13}^{R,Trig} + 6\hat{s}_{44}^{R,Trig}}. \tag{86}$$

2.6 Bounds for Monoclinic Symmetry

For monoclinic symmetry, there are thirteen independent Voigt and Reuss elastic constants.
Voigt elastic constants:
$\hat{c}_{11}^{V,Mon}, \hat{c}_{12}^{V,Mon}, \hat{c}_{13}^{V,Mon}, \hat{c}_{15}^{V,Mon}, \hat{c}_{22}^{V,Mon}, \hat{c}_{23}^{V,Mon}, \hat{c}_{25}^{V,Mon}, \hat{c}_{33}^{V,Mon}, \hat{c}_{35}^{V,Mon}, \hat{c}_{44}^{V,Mon}, \hat{c}_{46}^{V,Mon}, \hat{c}_{55}^{V,Mon}, \hat{c}_{66}^{V,Mon}$.

Reuss elastic constants:
$\hat{s}_{11}^{R,Mon}, \hat{s}_{12}^{R,Mon}, \hat{s}_{13}^{R,Mon}, \hat{s}_{15}^{R,Mon}, \hat{s}_{22}^{R,Mon}, \hat{s}_{23}^{R,Mon}, \hat{s}_{25}^{R,Mon}, \hat{s}_{33}^{R,Mon}, \hat{s}_{35}^{R,Mon}, \hat{s}_{44}^{R,Mon}, \hat{s}_{46}^{R,Mon}, \hat{s}_{55}^{R,Mon}, \hat{s}_{66}^{R,Mon}$.

These constants are used to construct bounds for monoclinic symmetry. The upper bounds are denoted by $k^{V,Mon}$, $G^{V,Mon}$, $\nu^{R,Mon}$, $E^{V,Mon}$. The lower bounds are denoted by $k^{R,Mon}$, $G^{R,Mon}$, $\nu^{V,Mon}$, $E^{R,Mon}$. The effective anisotropic elastic constants, k_{eff}^{Mon}, G_{eff}^{Mon}, ν_{eff}^{Mon} and E_{eff}^{Mon} of a monoclinic material must satisfy the following bounds:

$$k^{R,Mon} \le k_{eff}^{Mon} \le k^{V,Mon}, \qquad G^{R,Mon} \le G_{eff}^{Mon} \le G^{V,Mon}, \tag{87}$$

$$E^{R,Mon} \le E_{eff}^{Mon} \le E^{V,Mon}, \qquad \nu^{V,Mon} \le \nu_{eff}^{Mon} \le \nu^{R,Mon}. \tag{88}$$

k_{eff}^{Mon}, G_{eff}^{Mon}, ν_{eff}^{Mon} and E_{eff}^{Mon} are the best sets of elastic constants of a monoclinic material which should lie between the above bounds. The bulk modulus of Voigt for monoclinic symmetry can be obtained by substituting y, $\dot{z}$, z into the equation (5), this equation takes the form

$$k^{V,Mon} = \frac{y}{9} + \frac{2\dot{z}}{9}, \tag{89}$$

Where $y = \hat{c}_{11}^{V,Mon} + \hat{c}_{22}^{V,Mon} + \hat{c}_{33}^{V,Mon}$, $\dot{z} = \hat{c}_{12}^{V,Mon} + \hat{c}_{23}^{V,Mon} + \hat{c}_{13}^{V,Mon}$ and $z = \hat{c}_{44}^{V,Mon} + \hat{c}_{55}^{V,Mon} + \hat{c}_{66}^{V,Mon}$.
The shear modulus of Voigt for monoclinic symmetry can be obtained by substituting y,

$\dot{z}$, z into equation (6) then it becomes

$$G^{V,Mon} = \frac{y - \dot{z} + 3z}{15}. \tag{90}$$

Poisson ratio of Reuss for monoclinic symmetry is obtained by substituting the values of $k^{V,Mon}$ and $G^{V,Mon}$ in

$$\nu^{V,Mon} = \frac{1}{2}\left[1 - \frac{3G^{V,Mon}}{3k^{V,Mon} + G^{V,Mon}}\right], \tag{91}$$

Where $\nu^{V,Mon}$ represents the Poisson's ratio of Voigt for monoclinic symmetry and it is also expressed by putting the appropriate elastic constants of a monoclinic material

$$\nu^{V,Mon} = \frac{1}{2}\left[\frac{5y - 8\dot{z} - 2z}{6y - 9\dot{z} + z}\right]. \tag{92}$$

Young's modulus of Voigt for monoclinic symmetry is obtained by substituting the values of $k^{V,Mon}$ and $G^{V,Mon}$ in

$$E^{V,Mon} = \frac{27G^{V,Mon}k^{V,Mon}}{3G^{V,Mon} + 9k^{V,Mon}}, \tag{93}$$

Where $E^{V,Mon}$ represents the Young's modulus of Voigt for monoclinic symmetry and it is also expressed in terms of elasticity tensors as

$$E^{V,Mon} = \frac{(y + 2\dot{z})(y - \dot{z} + 3z)}{6y + 9\dot{z} + 3z}. \tag{94}$$

The bulk modulus of Reuss for monoclinic symmetry can be obtained by substituting $\hat{y}$, $\hat{w}$, $\hat{u}$ into the equation (11), and then it takes the form

$$k^{R,Mon} = \frac{1}{\hat{y} + 2\hat{w}}, \tag{95}$$

where $\hat{y} = \hat{s}_{11}^{R,Mon} + \hat{s}_{22}^{R,Mon} + \hat{s}_{33}^{R,Mon}$, $\hat{w} = \hat{s}_{12}^{R,Mon} + \hat{s}_{23}^{R,Mon} + \hat{s}_{13}^{R,Mon}$ and

$\hat{u} = \hat{s}_{44}^{R,Mon} + \hat{s}_{55}^{R,Mon} + \hat{s}_{66}^{R,Mon}$.

The shear modulus of Reuss for monoclinic symmetry can be obtained by substituting $\hat{y}$, $\hat{w}$, $\hat{u}$ into equation (12), then the equation becomes

$$G^{R,Mon} = \frac{15}{4\hat{y} - 4\hat{w} + 3\hat{u}} . \quad (96)$$

Poisson ratio of Reuss for monoclinic symmetry is obtained by substituting the values of $k^{R,Mon}$ and $G^{R,Mon}$ in

$$\nu^{R,Mon} = \frac{1}{2}\left[1 - \frac{3G^{R,Mon}}{3k^{R,Mon} + G^{R,Mon}}\right] . \quad (97)$$

Where $\nu^{R,Mon}$ represents the Poisson's ratio of Reuss for monoclinic symmetry and it is also expressed in terms of the appropriate elastic constants as

$$\nu^{R,Mon} = \frac{1}{2}\left[\frac{\hat{u} - 2\hat{y} - 8\hat{w}}{3\hat{y} + 2\hat{w} + \hat{u}}\right] . \quad (98)$$

Young's modulus of Reuss for monoclinic symmetry is obtained by substituting the values of $k^{R,Mon}$ and $G^{R,Mon}$ in

$$E^{R,Mon} = \frac{27G^{R,Mon}k^{R,Mon}}{3G^{R,Mon} + 9k^{R,Mon}} , \quad (99)$$

Where $E^{R,Mon}$ represents the Young's modulus of Reuss for monoclinic symmetry and it is also expressed by putting the appropriate elastic constants of a monoclinic material

$$E^{R,Mon} = \frac{15}{3\hat{y} + 2\hat{w} + \hat{u}} . \quad (100)$$

3 NUMERICAL EXAMPLES OF THE BOUNDS FOR VARIOUS TYPES OF MATERIAL SYMMETRIES

Numerical examples of the bounds for cubic, isotropic, tetragonal, transversely isotropic, trigonal, monoclinic, triclinic media are presented. Voigt elastic constants and Reuss elastic constants of selected materials are given for each corresponding anisotropic symmetry type. These data are used to compute the lower and upper bounds on the anisotropic elastic constants for all types of materials. Numerical bounds on the anisotropic elastic constants are calculated by MATLAB for all types of material symmetries. The units of Voigt elastic constant data and G^V, k^V, E^V, ν^V for all material symmetry types are GPa. The units of Reuss elastic constant data and G^R, k^R, E^R, ν^R for all material symmetry types are $(\text{TPa})^{-1}$.

3.1 For Cubic Media

Table 3.1:Voigt elastic constant data of cubic media

Cubic Media	$\hat{\mathbf{c}}_{11}^{V,Cub}$	$\hat{\mathbf{c}}_{12}^{V,Cub}$	$\hat{\mathbf{c}}_{44}^{V,Cub}$
Diamond,C (Grimsditch and Ramdas, 1975)	1040	170	550
Platinum(Pt)(Macfarlane,Rayne and Jones, 1965)	347	251	76.5
Beryllium oxide(BeO) (Martin, 1972)	381	147	200
Rubidium silver iodide($RbAg_4I_5$) (Graham and Chang, 1975)	16.5	9.34	4.89
Thallium manganese chloride($TIMnCl_3$) (Aleksandrov, Anistratov, Krupnyi, et al. 1975).	44.8	28.3	16.1

Table 3.2: Reuss elastic constant data of cubic media

Cubic Media	$\hat{\mathbf{s}}_{11}^{R,Cub}$	$\hat{\mathbf{s}}_{12}^{R,Cub}$	$\hat{\mathbf{s}}_{44}^{R,Cub}$
Diamond,C (Grimsditch and Ramdas, 1975)	1.01	-0.14	1.83
Platinum(Pt) (Macfarlane,Rayne and Jones, 1965)	7.35	-3.08	13.1
Beryllium oxide(BeO) (Martin, 1972)	3.35	-0.93	5.01
Rubidium silver iodide($RbAg_4I_5$) (Graham and Chang, 1975)	103	-37	204
Thallium manganese chloride($TIMnCl_3$) (Aleksandrov, Anistratov, Krupnyi, et al. 1975).	43.7	-16.9	62.1

Table 3.3: Lower bounds on cubic elastic constants

Cubic Media	$G^{R,Cub}$	$k^{R,Cub}$	$E^{R,Cub}$	$\nu^{V,Cub}$
Diamond,C	495.5	456.6	1091.7	0.099
Platinum(Pt)	61.7	280.1	172.5	0.393
Beryllium oxide(BeO)	155.5	223.7	378.8	0.203
Rubidium silver iodide($RbAg_4I_5$)	4.30	11.5	11.4	0.334
Thallium manganese chloride ($TIMnCl_3$)	11.7	33.7	31.4	0.330

Table3.4: Upper bounds on cubic elastic constants

Cubic Media	$G^{V,Cub}$	$k^{V,Cub}$	$E^{V,Cub}$	$\nu^{R,Cub}$
Diamond,C	504	460	1107.5	0.102
Platinum(Pt)	65.1	283	181.4	0.397
Beryllium oxide(BeO)	166.8	225	401.3	0.218
Rubidium silver iodide ($RbAg_4I_5$)	4.37	11.7	11.7	0.335
Thallium manganese chloride ($TIMnCl_3$)	12.96	33.8	34.5	0.345

3.2 For Isotropic Media

For some materials, it is possible to make approaches from cubic symmetry to isotropic symmetry. With cubic symmetry, three independent elastic constants are needed. If the medium is elastically isotropic, the elastic properties are independent of direction and only two independent elastic constants are required. These constants are ` $\hat{c}_{11}^{V,Iso}$ ' and ` $\hat{c}_{12}^{V,Iso}$ '. The relationship between the cubic and isotropic symmetry is `$2*\hat{c}_{44}^{V,Cub}=\hat{c}_{11}^{V,Cub}-\hat{c}_{12}^{V,Cub}$ '. By this equality, we can get the anisotropy ratio which is demostrated by A and $A=\frac{2\cdot\hat{c}_{44}^{V,Cub}}{\left(\hat{c}_{11}^{V,Cub}-\hat{c}_{12}^{V,Cub}\right)}$. A is unitless. The degree of anisotropy is measured by the deviation of A from the value $A=1$, characteristic of an isotropic medium. If the deviation from 1 is small, then we can say that the material is practically isotropic. Voigt and Reuss elastic constants of some nearly isotropic materials are given in Tables 3.5 and 3.6 respectively.

Table 3.5: Voigt elastic constant data of isotropic media

Isotropic Media	$\hat{c}_{11}^{V,Cub}$	$\hat{c}_{12}^{V,Cub}$	$\hat{c}_{44}^{V,Cub}$	A
Aluminium (Robrock and Schilling, 1976).	108	62	28.3	1.23
Alloy: Aluminium-magnesium at %7.7 Mg	103	57	29	1.26

(Gault, Boch and Dauger, 1977)				
Alloy: Aluminium-magnesium at %4.5 Mg (Gault, Boch and Dauger, 1977)	104	58	28.8	1.25
Alloy:Titanium-Vanadium at %53 V (Fisher, 1975).	177.3	114.7	41.3	1.32
Alloy:Lead-indium at %9 In (Madhava and Saunders, 1977)	59.70	33.70	13.90	1.10

Table 3.6: Reuss elastic constant data of isotropic media

Isotropic Media	$\hat{s}_{11}^{R,Cub}$	$\hat{s}_{12}^{R,Cub}$	$\hat{s}_{44}^{R,Cub}$
Aluminium (Robrock and Schilling, 1976).	16	-5.8	35.3
Alloy: Aluminium-magnesium at %7.7 Mg (Gault, Boch and Dauger, 1977)	16	-5.7	34.5
Alloy: Aluminium-magnesium at %4.5 Mg (Gault, Boch and Dauger, 1977)	16	-5.7	34.7
Alloy: Titanium-Vanadium at %53 V (Fisher, 1975).	11.5	-4.5	24.2
Alloy:Lead-indium at %9 In (Madhava and Saunders, 1977)	28.3	-10.2	71.9

Table 3.7: Lower bounds on isotropic elastic constants

Isotropic Media	$G^{R,Cub}$	$k^{R,Cub}$	$E^{R,Cub}$	$\nu^{V,Cub}$
Aluminium	26	75.8	69.8	0.348
Alloy: Aluminium-magnesium at %7.7 Mg	26.3	72.3	70.3	0.336
Alloy: Aluminium-magnesium at %4.5 Mg	26.2	72.5	70.1	0.339
Alloy:Titanium-Vanadium at %53 V	36.6	133.3	100.6	0.374
Alloy:Lead-indium at %9 In	13.5	42.2	36.7	0.355

Table 3.8: Upper bounds on isotropic elastic constants

Isotropic Media	$G^{V,Cub}$	$k^{V,Cub}$	$E^{V,Cub}$	$\nu^{R,Cub}$
Aluminium	26	77.3	70.7	0.349
Alloy: Aluminium-magnesium at %7.7 Mg	26.6	72.5	71.1	0.338
Alloy: Aluminium-magnesium at %4.5 Mg	26.5	73.3	70.9	0.339
Alloy:Titanium-Vanadium at %53 V	37.3	135.6	102.5	0.374
Alloy:Lead-indium at %9 In	13.54	42.4	36.7	0.355

3.3 For Tetragonal Media

Table 3.9: Voigt elastic constant data of tetragonal media

Tetragonal Media	$\hat{c}_{11}^{V,Tet}$	$\hat{c}_{12}^{V,Tet}$	$\hat{c}_{13}^{V,Tet}$	$\hat{c}_{33}^{V,Tet}$	$\hat{c}_{44}^{V,Tet}$	$\hat{c}_{66}^{V,Tet}$
Indium-cadmium alloy, In-3.42 at %Cd (Madhava and Saunders, 1977).	44.8	41	40.5	44.1	6.86	11.25
Ammonium dihydrogen arsenate (piezoel.), $NH_4H_2ASO_4$ (Haussü hl, 1964).	62.2	8.6	18.4	29.6	6.69	6.22
Zircon, $ZrSiO_4$ (metamict) (Özkan and Cartz, 1973).	284	73	119	309	77.5	47.7

Table 3.10: Reuss elastic constant data of tetragonal media

Tetragonal Media	$\hat{s}_{11}^{R,Tet}$	$\hat{s}_{12}^{R,Tet}$	$\hat{s}_{13}^{R,Tet}$	$\hat{s}_{33}^{R,Tet}$	$\hat{s}_{44}^{R,Tet}$	$\hat{s}_{66}^{R,Tet}$
Indium-cadmium alloy, In-3.42 at %Cd (Madhava and Saunders, 1977).	174	-86	-81	172	146	89
Ammonium dihydrogen arsenate (piezoel.), $NH_4H_2ASO_4$ (Haussü hl, 1964).	19.8	1.1	-13	50	149	161
Zircon, $ZrSiO_4$ (metamict) (Özkan and Cartz, 1973).	4.26	-0.49	1.45	4.36	12.9	21

Table 3.11: Lower bounds on tetragonal elastic constants

Tetragonal Media	$G^{R,Tet}$	$k^{R,Tet}$	$E^{R,Tet}$	$\nu^{V,Tet}$
Indium-cadmium alloy, In-3.42 at %Cd	3.6	41.7	10.4	0.434
Ammonium dihydrogen arsenate (piezoel.),$NH_4H_2ASO_4$	8.2	25.1	22.1	0.320
Zircon, $ZrSiO_4$ (metamict)	73	163.9	190.6	0.297

Table 3.12: Upper bounds on tetragonal elastic constants

Tetragonal Media	$G^{V,Tet}$	$k^{V,Tet}$	$E^{V,Tet}$	$\nu^{R,Tet}$
Indium-cadmium alloy, In-3.42 at %Cd	5.77	41.97	16.6	0.459
Ammonium dihydrogen arsenate (piezoel.),$NH_4H_2ASO_4$	11.16	27.2	29.5	0.353

Zircon, $ZrSiO_4$ (metamict)	78.3	166.6	203	0.306

3.4 For Transversely Isotropic Media

Table 3.13: Voigt elastic constant data of transversely isotropic media

Transversely Isotropic Media	$\hat{\mathbf{c}}_{11}^{V,Trans}$	$\hat{\mathbf{c}}_{12}^{V,Trans}$	$\hat{\mathbf{c}}_{13}^{V,Trans}$	$\hat{\mathbf{c}}_{33}^{V,Trans}$	$\hat{\mathbf{c}}_{44}^{V,Trans}$
Cobalt(Co) (Masumoto, Saito and Kikuchi,1967)	295	159	111	335	71
Hafnium (Hf) (Fisher and Renken, 1964).	181	77	66	197	55.7
Zinc (Zn) (Singh, Singh and Chendra, 1977).	165	31.1	50	61.8	39.6
Bone(dried phalanx) (Bonfield and Grynpas, 1977)	21.2	9.50	10.2	37.4	7.50
Polystyrene (Wright, Faraday and White et al., 2002).	5.20	2.75	2.75	5.70	1.30

Table 3.14: Reuss elastic constant data of transversely isotropic media

Transversely Isotropic Media	$\hat{\mathbf{s}}_{11}^{R,Trans}$	$\hat{\mathbf{s}}_{12}^{R,Trans}$	$\hat{\mathbf{s}}_{13}^{R,Trans}$	$\hat{\mathbf{s}}_{33}^{R,Trans}$	$\hat{\mathbf{s}}_{44}^{R,Trans}$
Cobalt(Co) (Masumoto, Saito and Kikuchi,1967)	5.11	-2.37	-0.94	3.69	14.1
Hafnium (Hf) (Fisher and Renken, 1964).	7.16	-2.48	-1.57	6.13	18
Zinc(Zn) (Singh, Singh and Chendra, 1977)	8.22	0.60	-7	27.7	25.3
Bone(dried phalanx) (Bonfield and Grynpas, 1977)	63	-23	-11	33	133
Polystyrene (Wright, Faraday and White et al., 2002).	299	-109	-91	264	770

Table 3.15: Lower bounds on transversely isotropic elastic constants

Transversely Isotropic Media	$G^{R,Trans}$	$k^{R,Trans}$	$E^{R,Trans}$	$\nu^{V,Trans}$
Cobalt(Co)	74.2	184.8	196.4	0.317
Hafnium(Hf)	55.4	108	142	0.280
Zinc(Zn)	35.1	57.7	87.5	0.236
Bone(dried phalanx)	7.04	14.5	18.2	0.290
Polystyrene	1.28	3.57	3.44	0.340

Table 3.16: Upper bounds on transversely isotropic elastic constants

Transversely Isotropic Media	$G^{V,Trans}$	$k^{V,Trans}$	$E^{V,Trans}$	$\nu^{R,Trans}$
Cobalt(Co)	78.3	187.4	206	0.323

Hafnium(Hf)	56	108.6	143.4	0.281
Zinc(Zn)	46.6	72.7	115.2	0.247
Bone(dried phalanx)	7.50	15.5	19.4	0.290
Polystyrene	1.29	3.62	3.46	0.340

3.5 For Trigonal Media

Table 3.17: Voigt elastic constant data of trigonal media

Trigonal Media	$\hat{\mathbf{c}}_{11}^{V,Trig}$	$\hat{\mathbf{c}}_{12}^{V,Trig}$	$\hat{\mathbf{c}}_{13}^{V,Trig}$	$\hat{\mathbf{c}}_{14}^{V,Trig}$	$\hat{\mathbf{c}}_{33}^{V,Trig}$	$\hat{\mathbf{c}}_{44}^{V,Trig}$
Antimony (Epstein and De Bretteville, 1965).	99.4	30.9	26.4	21.6	44.5	39.5
Magnesite, $MgCO_3$ (Humbert and Plique, 1972).	259	75.6	58.8	-19	156	54.8
Haematite, Fe_2O_3 (Subrahmanyam, 1958).	242	54.9	15.7	-12.7	228	85.3
As-Sb at % As 25.5 (Akgöz, Isci, Saunders, 1976)	106.7	48.4	28.5	18.8	48	40.8
Arsenic (Pace and Saunders, 1971).	130.2	30.3	64.3	-3.71	58.7	22.5

Table 3.18: Reuss elastic constant data of trigonal media

Trigonal Media	$\hat{\mathbf{s}}_{11}^{R,Trig}$	$\hat{\mathbf{s}}_{12}^{R,Trig}$	$\hat{\mathbf{s}}_{13}^{R,Trig}$	$\hat{\mathbf{s}}_{14}^{R,Trig}$	$\hat{\mathbf{s}}_{33}^{R,Trig}$	$\hat{\mathbf{s}}_{44}^{R,Trig}$
Antimony (Epstein and De Bretteville, 1965).	16.2	-6.1	-5.9	-12.2	29.5	38.6
Magnesite, $MgCO_3$ (Humbert and Plique, 1972).	4.67	-1.22	-1.30	2.04	7.41	19.7
Haematite, Fe_2O_3 (Subrahmanyam, 1958).	4.41	-1.02	-0.23	0.79	4.43	11.9
As-Sb at % As 25.5 (Akgöz, Isci, Saunders, 1976)	15.4	-6.96	-4.96	-9.76	27	33.3
Arsenic (Pace and Saunders, 1971).	30.3	20.2	-55.2	1.67	137.8	45

Table 3.19: Lower bounds on trigonal elastic constants

Trigonal Media	$G^{R,Trig}$	$k^{R,Trig}$	$E^{R,Trig}$	$\nu^{V,Trig}$
Antimony	21.9	38.3	74.4	0.207
Magnesite, $MgCO_3$	63.6	109.8	159.9	0.245

Haematite,Fe_2O_3	92.1	97.2	209.9	0.136
As-Sb at % As 25.5	23.7	41.6	59.8	0.243
Arsenic	10.1	55.6	28.6	0.316

Table 3.20: Upper bounds on trigonal elastic constants

Trigonal Media	$G^{V,Trig}$	$k^{V,Trig}$	$E^{V,Trig}$	$\nu^{R,Trig}$
Antimony	33.3	45.6	80.3	0.260
Magnesite,$MgCO_3$	72.3	117.8	180.1	0.257
Haematite,Fe_2O_3	94.5	98.3	214.8	0.140
As-Sb at % As 25.5	32.6	52.5	80.9	0.261
Arsenic	29.7	70.8	78.1	0.414

3.6 For Monoclinic Media

Table 3.21: Voigt elastic constant data of monoclinic media

Monoclinic Media	**Coesite,SiO_2** (Weidner and Carleton, 1977).	**Diphenyl,$C_{12}H_{10}$**
$\hat{\mathbf{c}}_{11}^{V,Mon}$	161	5.95
$\hat{\mathbf{c}}_{12}^{V,Mon}$	82	4.05
$\hat{\mathbf{c}}_{13}^{V,Mon}$	103	2.88
$\hat{\mathbf{c}}_{15}^{V,Mon}$	-36	0.40
$\hat{\mathbf{c}}_{22}^{V,Mon}$	230	6.97
$\hat{\mathbf{c}}_{23}^{V,Mon}$	36	6.11
$\hat{\mathbf{c}}_{25}^{V,Mon}$	3	0.94
$\hat{\mathbf{c}}_{33}^{V,Mon}$	232	14.6
$\hat{\mathbf{c}}_{35}^{V,Mon}$	-39	2.02
$\hat{\mathbf{c}}_{44}^{V,Mon}$	67.8	1.83
$\hat{\mathbf{c}}_{46}^{V,Mon}$	10	-0.89
$\hat{\mathbf{c}}_{55}^{V,Mon}$	73.3	2.26
$\hat{\mathbf{c}}_{66}^{V,Mon}$	58.8	4.11

Table 3.22: Reuss elastic constant data of monoclinic media

Monoclinic Media	**Coesite, SiO_2** (Weidner and Carleton, 1977).	**Diphenyl, $C_{12}H_{10}$**
$\hat{s}_{11}^{R,Mon}$	11.3	283
$\hat{s}_{12}^{R,Mon}$	-3.50	-184
$\hat{s}_{13}^{R,Mon}$	-3.90	19
$\hat{s}_{15}^{R,Mon}$	3.60	9
$\hat{s}_{22}^{R,Mon}$	5.30	346
$\hat{s}_{23}^{R,Mon}$	0.40	-107
$\hat{s}_{25}^{R,Mon}$	-1.70	-16
$\hat{s}_{33}^{R,Mon}$	6.20	118
$\hat{s}_{35}^{R,Mon}$	1.40	-65
$\hat{s}_{44}^{R,Mon}$	15.1	611
$\hat{s}_{46}^{R,Mon}$	-2.60	132
$\hat{s}_{55}^{R,Mon}$	16.2	509
$\hat{s}_{66}^{R,Mon}$	17.4	272

Table 3.23: Lower and upper bounds on monoclinic elastic constants

Monoclinic Media	**Coesite, SiO_2**	**Diphenyl, $C_{12}H_{10}$**
$\mathbf{G}^{R,Mon}$	56.5	1.82
$\mathbf{G}^{V,Mon}$	66.8	2.61
$\mathbf{k}^{R,Mon}$	113.6	4.93
$\mathbf{k}^{V,Mon}$	118.3	5.96
$\mathbf{E}^{R,Mon}$	145.5	4.86
$\mathbf{E}^{V,Mon}$	168.6	6.82
$\nu^{R,Mon}$	0.287	0.336
$\nu^{V,Mon}$	0.263	0.309

3.6 For Triclinic Media

Table 3.24: Voigt elastic constant data of triclinic media

Triclinic Media	**Ammonium tetroxalate dehydrate** (Küppers and Siegert, 1970).	**Potassium tetroxalate dehydrate** (Küppers and Siegert, 1970).
$\hat{c}_{11}^{V,Tric}$	21.9	25.4
$\hat{c}_{12}^{V,Tric}$	12	11.8
$\hat{c}_{13}^{V,Tric}$	10.4	9.83
$\hat{c}_{14}^{V,Tric}$	1.60	0.72
$\hat{c}_{15}^{V,Tric}$	6	6.12
$\hat{c}_{16}^{V,Tric}$	-1	-1.23
$\hat{c}_{22}^{V,Tric}$	45.9	47.8
$\hat{c}_{23}^{V,Tric}$	16.3	14
$\hat{c}_{24}^{V,Tric}$	11.6	11.3
$\hat{c}_{25}^{V,Tric}$	2	1.46
$\hat{c}_{26}^{V,Tric}$	-3.80	-2.70
$\hat{c}_{33}^{V,Tric}$	36.4	34.3
$\hat{c}_{34}^{V,Tric}$	3.8	2.19
$\hat{c}_{35}^{V,Tric}$	2	1.47
$\hat{c}_{36}^{V,Tric}$	-0.8	0.40
$\hat{c}_{44}^{V,Tric}$	10.4	10.2
$\hat{c}_{45}^{V,Tric}$	0.10	-0.82
$\hat{c}_{46}^{V,Tric}$	0.10	0.53
$\hat{c}_{55}^{V,Tric}$	5.40	5.69
$\hat{c}_{56}^{V,Tric}$	0.10	0.70
$\hat{c}_{66}^{V,Tric}$	4.44	4.99

Table 3.25: Reuss elastic constant data of triclinic media

Triclinic Media	**Ammonium tetroxalate dehydrate** (Küppers and Siegert, 1970).	**Potassium tetroxalate dehydrate** (Küppers and Siegert, 1970).
$\hat{s}_{11}^{R,Tric}$	81.9	66.2
$\hat{s}_{12}^{R,Tric}$	−15.2	−10.2
$\hat{s}_{13}^{R,Tric}$	−13	−12.4
$\hat{s}_{14}^{R,Tric}$	9.80	2.8
$\hat{s}_{15}^{R,Tric}$	−80.7	−67.6
$\hat{s}_{16}^{R,Tric}$	5.80	20.9
$\hat{s}_{22}^{R,Tric}$	42	37.4
$\hat{s}_{23}^{R,Tric}$	−9.80	−9.80
$\hat{s}_{24}^{R,Tric}$	−41.1	−40.4
$\hat{s}_{25}^{R,Tric}$	5.30	−4.80
$\hat{s}_{26}^{R,Tric}$	31.7	23.5
$\hat{s}_{33}^{R,Tric}$	35.2	36.1
$\hat{s}_{34}^{R,Tric}$	0.20	5.50
$\hat{s}_{35}^{R,Tric}$	4.90	8.90
$\hat{s}_{36}^{R,Tric}$	−5.40	−13.1
$\hat{s}_{44}^{R,Tric}$	140	146
$\hat{s}_{45}^{R,Tric}$	1.60	32.2
$\hat{s}_{46}^{R,Tric}$	−37.5	−41.6
$\hat{s}_{55}^{R,Tric}$	271	259
$\hat{s}_{56}^{R,Tric}$	−20.9	−59.5
$\hat{s}_{66}^{R,Tric}$	254	232

Table 3.26: Lower and upper bounds on triclinic elastic constants

Triclinic Media	Ammonium tetroxalate dihydrate	Potassium tetroxalate dihydrate
$G^{R,Tric}$	5.39	5.77
$G^{V,Tric}$	8.41	8.97
$k^{R,Tric}$	12	13.4
$k^{V,Tric}$	20.2	19.9
$E^{R,Tric}$	14.1	15.1
$E^{V,Tric}$	22.2	23.4
$\nu^{R,Tric}$	0.317	0.311
$\nu^{V,Tric}$	0.305	0.304

4 DISCUSSION OF RESULTS

Results of the tables illustrated in section 3 can be interpreted as follows:

For cubic symmetry:

According to bounds presented in Tables 3.3 and 3.4, it is obvious to see that Rubidium silver iodide is close to isotropy more than the other cubic materials. Since the intervals between Voigt and Reuss bounds on elastic constants of the material are very closer, the effective elastic constants are selected from a narrow range. On the other hand, Beryllium oxide is a kind of compound and exhibits the most anisotropy among the other cubic materials. Since the elastic constants of it, especially the values of Young's modulus of Voigt and Reuss for these materials are considerably different, it's mechanical and elastic behavior are expected to be more anisotropic.

For tetragonal symmetry:

From Tables 3.11, 3.12, it is seen that Indium-cadmium alloy composed of % 3.42 Cadmium and % 96.58 Indium, is close to isotropy more than the others. As a result, the interval between bounds on Reuss and Voigt bulk modulus for the alloy is very small. Whereas Zircon shows the most anisotropy among the other materials.

For transversely isotropic symmetry:

According to calculated bounds in Tables 3.15 and 3.16, Hafnium, Bone (dried phalanx) and Polystyrene are close to isotropy more than the other transversely isotropic materials. Because Voigt and Reuss elastic constants of them for Voigt and Reuss notation are very closer, Reuss and Voigt Poisson's ratio of these materials are the same.

On the other hand, Zinc exhibits the most anisotropy. Since the elastic constants, especially the values of Young’s modulus of Voigt and Reuss for Zinc are considerably different, its mechanical and elastic behavior are expected to be more anisotropic. As a result, the effective elastic constants of Zinc are selected from a large range.

For trigonal symmetry:
According to Tables 3.19 and 3.20, it is observed that Haematite is close to isotropy more than the other trigonal materials. Since the intervals between the corresponding Voigt and Reuss bounds on effective elastic constants for Haematite are smaller than the others, effective elastic constants of Haematite can be selected from a smaller range than the other trigonal materials. Moreover, Magnesite, Antimony, As-Sb alloy (which is composed of % 25.5 Arsenic and % 74.5 Antimony) show anisotropy due to the corresponding values of Tables 3.19 and 3.20. While Arsenic has the greatest anisotropy among them because of the large intervals between Reuss and Voigt bounds. It is seen that mechanical and elastic behavior of Arsenic are more anisotropic than other trigonal materials.

For monoclinic symmetry:
The intervals between Reuss and Voigt bounds on effective elastic constants (given in Table 3.23) for Diphenyl are closer than Coesite. As a result, it can be said that Diphenyl is close to isotropy more than Coesite.

For triclinic symmetry:
If Table 3.26 is examined in detail, it is seen that the anisotropy is the highest for triclinic materials. The intervals between the corresponding Reuss and Voigt bounds are very large. So the effective anisotropic elastic constants are selected from a large range.
These results show that the materials selected from same anisotropic elastic symmetry, depending upon the size of intervals between Reuss and Voigt bounds, except triclinic symmetry can exhibit whether close to isotropy or anisotropy.

5 CONCLUSION

In this paper, it has been shown that it is possible to construct bounds on the anisotropic elastic constants of any anisotropic elastic symmetry in terms of elasticity and compliance tensors. Specific bounds have been presented for the anisotropic elastic constants of cubic, isotropic, transversely isotropic, tetragonal, trigonal symmetries. It has been mainly focussed on engineering elastic properties of selected materials and represented anisotropy in terms of engineering properties: k , G , and ν , E in order to construct bounds. Constructing bounds on the anisotropic elastic constants provides a deeper understanding about mechanical behavior of anisotropic materials. It also has significant effects on many applications in different fields such as:
1) design of textured (non-crystalline) materials,
2) examining the material symmetry types in detail,
3) determination of materials which are highly anisotropic or close to isotropic.

5 REFERENCES

Hill, R., (1952). The elastic behaviour of a crystalline aggregate, Proceedings of the Physical Society of London, A 65, 349–354.

Hill, R., (1963). Elastic properties of reinforced solids: some theoretical principles, Journal of the Mechanics and Physics of Solids, 11, 357–372.

Dinçkal, Ç. (2011). Analysis of elastic anisotropy of wood material for engineering applications, Journal of Innovative Research in Engineering and Science 2(2), April. pp. 67-80.

Reuss, A., (1929). Berechnung der fliebgrenze von mischkristallen auf grund der plastizitä tsbedingung für einkristalle, ZAMM, 9, 49–58.

Voigt, W., (1928). Lehrbuch der kristallphysik, Teubner, Leipzig.

Hearmon, R.F.S., (1961). An Introduction to Applied Anisotropic Elasticity, Oxford University Press, 7- 44.

Nye, J.F., (1957). Physical Properties of Crystals; their representation by tensors and matrices. Oxford: At the Clarendon Press.

Grimsditch, M.H. and A.K.Ramdas (1975). Brillouin-scattering in Diamond, Physical Review B;Condensed Matter and Materials Physics, 11, 3139-3148.

Macfarlane, R.E., Rayne, J.A., and C.K. Jones, (1965). Anomalous temperature dependence of shear modulus C44 for Platinum, Physics Letters, 18, 91.

Martin, R.M. (1972). Relation between elastic tensors of Wurtzite and Zincblende structure materials, Physical Review, B 6, 4546-4553.

Graham, L.J.and R. Chang (1975). Temperature and pressure-dependence of elastic properties of RBAG4I5, Journal of Applied Physics, 46, 2433-2438.

Aleksandrov, K.S., Anistratov, A.T., Krupnyi, et al. (1975). X-Ray optical and ultrasound studies of phase-transitions in TLMNCL3, Fizika Tverdogo Tela, 17, 735-740.

Robrock, K.H. and W.Schilling (1976). Diaelastic modulus change of aluminium after low-temperature electron-irradiation, Journal of Physics F-Metal Physics, 6, 303-314.

Gault, C., Boch, P. and A. Dauger (1977). Variations of elastic-constants of Aluminium-Magnesium single-crystals with Guinier-Preston Zones, Physica Status Solidi A-Applied Research, 43, 625-632.

Fisher, E.S., (1975). A review of solute effects on the elastic modulus of bcc transition in Physics of Solid Solution Strengthening. Proc. Symposium 1973 eds. E.W. Collings and H.L. Gegel., New York, Plenum Press, 195.

Van Der Planken, J., Greiner, J.D. and Smith, J.F., (1971). Magnetic susceptibilities and single crystalline elastic constants of Lead-Indium alloys, Journal of Materials Science., 6, 1331.

Madhava, M.R., and Saunders, G.A.,(1977). Ultrasonic study of elastic phase-transition in In-Cd alloys, Philosophical Magazine , 36, 777-796.

Haussühl, S., (1964). Elastische und thermoelastische eigenschaften von KH_2 PO_4 KH_2 ASO_4 NH_4 H_2 PO_4 NH_4 H_2 ASO_4 und RBH_2 PO_4 , Zeitschrift fur Kristallographie, 120, 401.

Özkan, H. and L. Cartz (1973). AIP Conference; AIP Conference Proceedings, 1974, 17, 21.

Masumoto, H., Saito, H. and Kikuchi, M., (1967). Thermal expansion and temperature dependence of Young's Modulus of single crystals of hexagonal, Cobalt, Science Reports of the Research Institutes Tohoku University A- Physics Chemistry and Metallurgy, 19, 172.

Fisher, E.S., Renken, C.J., (1964). Single-crystal elastic moduli + HCP to BCC transformation in TI, ZR, +HF, Physical Review A-General Physics, 135, 482.

Singh, D.P., Singh, S. and Chendra S., (1977). Indian J. Phys., A 51, 97.

Bonfield, W., Grynpas, M.D., (1977). Anisotropy of Young's modulus of Bone, Nature London, 270, 453-454.

Wright, A., Faraday, C.S.N., White, E.F.T., et al., (2002). Elastic constants of oriented glassy polymers, Journal of Physics D- Applied Physics, 1971, 4.

Epstein, S. and De Bretteville, A.P., (1965). Elastic constants and wave propagation in Antimony and Bismuth, Physical Review, 138, A771.

Humbert, P. and Plique, F., (1972). Elastic properties of monocrystalline rhombohedral carbonates- Calcite, Magnesite, Dolomite; Comptes Rendus Hebdomadaines des Sé ances de l Academie des Sciences Serie B, 275, 391.

Subrahmanyam, S.V., (1958). Proc. Indian Acad. Sci., A 47, 25.

Akgö z, Y.C., Isc, C., Saunders, G.A., (1976). The elastic constants of antimony-25.5 at. % arsenic alloy single crystals, Journal of Materials Science, 11, 291-296.

Pace, N.G. and Saunders G.A., (1971). Elastic wave propagation in group-VB semimetals, Journal of Physics and Chemistry of Solids, 32, 1585.

Weidner, D.J. and Carleton, H.R., (1977). Elasticity of Coesite, Journal of Geophysical Research, 82, 1334-1346.

Küppers, H. and Siegert, H., (1970). Elastic constants of triclinic crystals Ammonium and Potassium Tetroxalate Dihydrate, Acta Crystallographica Section A, 26, 401.