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Alternative solution of delta–function potential by green functions and renormalization

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Abstract

Simple technique of constructing the Green's function that is associated with the Hamiltonian, $H = Ho + \lambda\delta(x)$ is presented. The result was applied in a case where H_o is the Hamiltonian of a free particle in given R dimensions. The theoretical concepts of field such as regularization and renomalization was introduced in order to handle the case at infinity which characterized the formal Green's function for R ≥ 2. A comparison was also made with alternative derivation using time –independent Schrodinger equation for a particle in potential, $V_{(x)} = \lambda\delta(x)$.

1. Introduction

Green's functions play a vital role in the solution of concepts involving linear ordinary and partial differential equations. Also, it is employed in the study of electromagnetic waves, wave propagating in a spatially inhomogenous medium (Ugwu et al., 2005). In quantum electrodynamics just like most quantum field theories is plagued with infinities of which the meaningful information can only be extracted from it no matter the nature of such infinities if the Green's function is employed. Such infinities can be dealt with in two stages; by

a. Introducing a cut-off which gives rise to finite answers and
b. Redefining the parameters of the theory in order to absorb the divergences, which may appear when the cut-off is ignored. The steps that are involved are regularization and renormalization of the theory. In non-relativistic quantum mechanics, this sort of infinities occur if the potential is singular for instance the Dirac delta-function potential in two, or more dimensions (Albeverio et al., 1988; Jackiw, 1991; Goldzinky et al., 1991; Abaronov-Bohrn potential in Manual et al., 1994; Park, 1995). This provides a unique framework in which the important concept of renorma-lization can be explained free from the technical complication and approach as normally obtained in quantum field theory.

In this case, we solve the Dirac delta-function potential using Green's function. This concept has been studied using some other techniques such as exact solution of the Schrodinger equation (Goldzinky et al., 1991; Fernando et al., 1991) or integral version, the Lippmann-Scwinger equation in Mead et al, self-adjoint extension method of Albeverio et al (1998) and Jackiw (1991); Green's function technique of Park, (1995) and Cavalcantic et al., (1998). We are employing the latter, which have a closer resemblance with the techniques usually used in quantum field theory. Apart from this, it is more convenience to find the Green's function associated with unpertur-bed Hamlitonian, of the form $H = H_o + K_o\delta(x)$ when the Green's function associated with H_o is known. The result is applied to the case in which H_o is the Hamiltonian of a free particle in one, two and three dimensions. In some cases, infinity appears in the formal expression of the Green's function, although in two and three-dimensional cases, this infinity can be removed in consistent way by application of regularization and renormalization.

2. Green's functions

For delta-function potential

The Green's functional, G(E; x, x') associated with the Hamiltonian H is a solution to the differential equation given as

$$(E - H)\, G(E; x, x') = \delta(x - x') \qquad (1)$$

and satisfies the boundary condition.

$$\lim_{|x\text{-}x|\to\infty} G(E; x, x) = 0 \qquad (2)$$

here x and x' are points in Euclidean space with the corresponding δ(x –x') Dirac delta-function.

Using the completeness of the eigen functions of H to write the solution of equation 1 as

$$G(E;x,x^{'}) = \sum \frac{\psi_n(x)\psi_n(x^{'})}{E-E_n} \quad (3)$$

(Butkov, 1968) where E_n and φ_n are the eigenvalues and eigen-functions of H, respectively. If the Hamiltonian is written as the sum of two terms,

$$H = H_o + \lambda\delta(x) \quad (4)$$

and Green's function associated with the H_o is known, then it can be shown that there is a simple way to writing G in terms of the G_o. To achieve this, equation 1 is written in an integral form, Economou, (1979) neglecting the dependence of G on E as

$$G(x_1 x^{'}) = G_o((x_1 x^{'}) + \int d^R x^{'} G_o(x_1 x^{'})\lambda\delta(x^{'})G(x_1 x^{'}) \quad (5)$$

$$G(x_1 x^{'}) = G_o(x_1 x^{'}) + \lambda G_o(x,0)G(0_1 x^{'}) \quad (5)$$

Considering x =0 in the expression, we solve it for G(0,x') and then inserting the result in (5) the explicit expression for the Green's function associated with H, is expressed as;

$$G(x_1 x^{'}) = G_o(x_1 x^{'}) + \frac{G_o(x,0)G(0,x^{'})}{\frac{1}{\lambda} - G_o(0,0)} \quad (6)$$

it can be observed that the successive applications of this technique allows one to obtain the Green's functions for a potential with an arbitrary number of delta-functions.

3. Regularization and renormalization

To investigate the bound states of the Hamiltonian written in (4) where H_o is the Hamiltonian of a free particle in given dimensions, we assume that h=2m =1 and

$$H_o = -\Delta^2 = -\sum_{i=1}^{R} \frac{\partial^2}{\partial x^2} - \sum_{i=1}^{R} \frac{\partial^2}{\partial x_j^2} \quad (7)$$

Cavacantic et al (1998).

Considering from (3) that the energy levels of the states are poles of the Green's function. As there are in the problem, such poles can only appear as zero of the denominator of the second (6). Fourier transform was applied in (1) to enable us obtain $G_o(E_i$ x, x') where H was replaced with H_o.

$$G_o(E_i x_1 x') = \int \frac{d^R k}{(2\pi)^R} \frac{e^{ik}(x-x')}{E-k^2} \quad (8)$$

Where $I^2 = -E_c$ is the bound energy which leads to

$$\frac{1}{\lambda} + \int \frac{d^R k}{(2\pi)R} \frac{1}{l^2+k^2} = 0 \quad (9)$$

can be analyzed for different values of R.

Case 1, R =1, the integral in (9) becomes

$$\frac{1}{\lambda} + \int_{-\infty}^{\infty} \frac{dK}{2\pi R} \frac{1}{l^2+K^2} = 0 \quad (10)$$

$$\frac{1}{\lambda} + \frac{1}{2k} = 0 \quad (11)$$

$$\Rightarrow k = -\frac{\lambda}{n} \, and \; E_o = \frac{\lambda^2}{4} \quad (12)$$

An important point here is that λ must be considered negative for K to be positive since positive K means that the potential must be attractive in order to create a bound state.

Also time-independent Sch-rodinger equation for a particle in the potential $V_{(x)} = \lambda\delta(x)$ is given as

$$-\frac{d^2}{dx^2}\psi(x) + \lambda\delta(x)\psi(x) \quad (13)$$

And can be used to make alternative comparison. For x ≠0, (3) is of free particle which can be solved for $E - K^2 < 0$ that if continuity at the origin is assumed,

$$\psi(x) = A\exp - K(x) \quad (14)$$

Integrating (13) from -ε to ε letting ε → 0 with restriction on the possible values of K;

$$-\psi'(0^+) + \psi'(0^-) + \lambda\psi(0) = 0 \quad (15)$$

Which gives $k = \frac{\lambda}{2}$

This expression is the same as that in (12) which expresses an indication that alternative method, which is more elementary in nature can be used in this problem.

Case 2

In this case, G_o (E; 0,0) is divergent and this has to be dealt with by introducing a cut-off in the integral of equation and absorbing the dependence on the cut-off by redefining the parameters in the theory (the "coupling constant" λ). This procedure is called regularization and renormalization. The first step is to regularize the integral:

$$\int \frac{d^2k}{(2\pi 2^2}\frac{1}{l^2+k^2} = \frac{1}{2\pi}\int_n^\infty \frac{kdk}{l^2+k^2} = \frac{1}{4\pi}\ln\left(\frac{L^2}{K^2}+1\right) \quad (17)$$

Another step to redefine the coupling constant λ as λ_c in order to absorb the divergent part as

$$\frac{1}{\lambda_c} \equiv \frac{1}{\lambda} + \frac{1}{4\pi} In\left(\frac{L^2}{m^2}\right) \quad (18)$$

Where m is arbitrary parameter to keep the argument of the logarithm dimensionless. As L → ∞, the varying bare coupling constant λ is presented in a such away that the renormalized one λ_c remains finite and then (18) becomes

$$\frac{1}{\lambda_c} - \frac{1}{4\pi} In\left(\frac{K^2}{m^2}\right) = 0 \quad (19)$$

and the energy of the bound state becomes

$$E_c = -K^2 = -m^2 \exp\left(\frac{4\pi}{\lambda c}\right) \quad (20)$$

from (20) we observe that the Hamiltonian contains one parameter λ, but the energy obtained depended on two poarameters, λ_c and μ.

Regularizing (6) enable us to obtain the expression

$$\frac{1}{\lambda} - G_o(E;0,0) = \frac{1}{\lambda} + \frac{1}{2\pi}\int_0^1 \frac{kdk}{k^2-E}$$

$$= \frac{1}{\lambda} + \frac{1}{4\pi} In\left(\frac{L^2-E}{-E}\right) \quad (21)$$

which shows that the Green's function depends on a just one prameter apart from E, x and x^1. Similarly from (16) to (18),

$$\lambda^{-1} = -\frac{1}{4\pi} In\left(\frac{-L^2}{E_o}\right) \quad (22)$$

and substituting (22) in (21) and taking the limit L →∞,

$$\frac{1}{\lambda} - G_o(E;0,0) = -\frac{1}{4\pi}\left[In\left(\frac{E}{E_c}\right)\right] \quad (23)$$

this process is referred to as dimensional transmutation. It is observed that having started with a Hamiltonian with dimensionless parameters (the coupling constant, λ), theory containing dimensionful parameter (E_c) is arrived at. This was because in the process of renormalization, the dimensionful parameter μ was introduced to break the invariance scale of the theory.

4. Conclusion

Dirac delta-function potential had been studied using Green's function technique and renormalization, during which it was observed that the successive applications of the technique allows one to find the Green's function for a potential with an arbitrary number of delta-function. The technique agreed with that obtained using time-independent Schrodinger equation in the potential V(x)=λδ (x). The problem of divergence in G_0 (E; 0, 0) when R ≥ 2 was eliminated by introduction of cut-off in the integral and the use of regularization and renormalization as tools for absorbing the dependence on the cut-off.

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