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Stodola-Vianello Method for the Buckling Load Analysis of Euler-Bernoulli Beam on Pasternak Foundation

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Abstract

The determination of critical buckling loads of thin beam on Pasternak foundations (BoPF) subjected to in-plane compressive loads is vital to their analysis and design. This study presents the Stodola-Vianello iterative method for formulating and solving the governing ordinary differential equation (ODE) subject to the boundary conditions. The governing boundary value problem is expressed in iterative form using the method of four successive integrations after re-arranging the ODE. A suitable buckling mode is employed in the derived Stodola-Vianello iteration formula for the pinned-pinned end conditions studied. The convergence requirement of the *n*th buckling mode is used to derive the characteristic buckling equation whose roots are used to obtain the buckling loads at the *n*th buckling mode. The obtained expression for the *n*th buckling mode was found to be exact because exact buckling eigenfunction was used in the derivation. The critical buckling load was found to be exact and correspond to first buckling mode. The critical buckling load expression was expressed in standard form in terms of critical buckling load coefficients which was found to depend upon the beam parameter and parameters of the Pasternak foundation. The values of the critical buckling load coefficients were found to be in close agreement with previous studies. Exact buckling load solutions were obtained for all the buckling modes of the BoPF, and the critical buckling load was found to be identical with exact critical buckling load solutions obtained by previous researchers.

Keywords: Stodola-Vianello iteration method, critical buckling load, beam on Pasternak foundation, eigenfunction, eigenvalue, critical buckling load coefficient

1. Introduction

Many important problems in geotechnical engineering are idealized as beam on elastic foundation problems. The examples include buried pipelines and shallow footings. Two main theories of beams used are Euler-Bernoulli beam theory (EBBT) and Timoshenko beam theory (TBT). More advanced beam theories which consider shear deformation have been proposed and studied by Levinson (1981), Dahake and Ghugal (2013), Sayyad and Ghugal (2011) and others. Some elastic foundation models that have been used are Winkler, Pasternak, Vlasov, Hetenyi and Kerr. The classical thin beam theory also called EBBT, assumes that cross-sectional planes normal to the longitudinal axis of the beam remain plane and normal to the longitudinal axis after deformation. The assumption implies that the theory is unsuitable for the analysis of thick and moderately thick beams where shear deformation plays critical role in the behaviour. The EBBT however gives accurate results for thin beams because shear deformation effects are insignificant in their behaviour (Ike, 2018a).

Timoshenko beam theory was formulated by relaxing the assumption of orthogonality of plane cross-sections, and hence considering shear deformation. TBT is thus suitable for moderately thick and thick beams (Ike, 2019). Winker model is a discrete one-parameter elastic foundation model which represents the soil reaction pressure using an analogue representation of closely spaced, mutually independent linear elastic vertical springs with stiffness that is directly proportional to the beam deflection at the point. Its major disadvantage is the lack of continuity resulting in the inability to account for shear interaction between adjoining springs. Several attempts to overcome the limitations of the one-parameter discrete elastic modelled researchers like Pasternak, Vlasov and Hetenyi to develop two-parameter discrete elastic foundation models. In such two-parameter models the first parameter represents the vertical spring stiffness while the second parameter represents the coupling or interaction effects of the springs. Beams resting on elastic foundations and under the action of compressive forces can fail by buckling when such compressive forces reach some critical values. The vertical buckling load solution of the beam on elastic foundation is thus a vital aspect of the analysis and design. Hetenyi (1946) pioneered studies on the stability analysis of beam on elastic foundation using the classical mathematical methods. He determined exact critical buckling loads for thin beam resting on Winkler foundations for various end support conditions.

Timoshenko and Gere (1985) and later on Wang et al (2005) also investigated the stability problems of beam on elastic foundations. They solved the governing ODEs using mathematical methods for solving ODEs and derived exact critical buckling load solutions for uniform simply supported Euler-Bernoulli beams resting on elastic foundations. Hassan (2018) studied the stability of beam on elastic foundations under various end restraints. Atay and Coskun (2009) have applied the variational iteration method (VIM) to find the critical buckling loads of beam on elastic foundations under various end restraints. Taha (2014) used the recursive differentiation method (RDM) to solve stability problems of beam on elastic foundations. Mama et al (2020) used quintic polynomial shape functions in the finite element method to determine elastic buckling loads of beam on Winkler foundation. Taha and Hadima (2015) applied the recursive differentiation method to find critical buckling load solutions of variable cross-section beams on elastic foundation. Aristizabel-Ochoa (2013) studied the stability of beams on elastic foundations for various end restraints. Anghel and Mares (2019) have applied collocation techniques to solve stability problems of beam on elastic foundation. The collocation principle sought to obtain the solution to the governing boundary value problem (BVP) only at the collocation points leading to an approximate solution, and a reduction of the complex integration problem to an algebraic formulation. Soltani (2020) applied the finite element method to derive stability solutions to Timoshenko beam on elastic foundation.

Ike (2018b) applied finite integral transform using sinusoidal kernel functions to solve the eigenvalue problem of naturally vibrating Euler-Bernoulli beam resting on one-parameter discrete elastic foundations for Dirichlet boundary conditions. The sinusoidal kernel functions of the finite sine transform method satisfies the Dirichlet boundary conditions, and the BVP was converted to an algebraic eigenvalue problem which was solved to obtain exact eigenvalues from which the natural frequencies were found. Ofondu et al (2018) have applied the Stodola-Vianello iteration method to derive approximate but accurate critical buckling load solutions for Euler columns. They derived the iteration formula by successive integrations of the re-arranged Euler column buckling differential equation. They derived an algebraic buckling shape coordinate function for the clamped-pinned end conditions considered and used it in the iteration formula for the determination of successive iterates for the buckling shape function. They found that a few iterative steps resulted in accurate critical buckling load. Literature reveals that the Stodola-Vianello iteration method has not been applied to the stability analysis of thin beam on Pasternak foundation.

In this paper the Stodola-Vianello iterative method is applied to derive buckling load solutions for Euler-Bernoulli beam resting on Pasternak foundations for the case of Dirichlet boundary conditions.

2. Governing Equation/Theory

The governing equation for the buckling load problem of a thin beam on Pasternak foundation is given by:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2v(x)}{dx^2}\right)+P\,\frac{d^2v(x)}{dx^2}+k_1v(x)-k_2\,\frac{d^2v(x)}{dx^2}=q(x) \qquad (1)$$

where x is the longitudinal coordinate axis of the beam,
$v(x)$ is the transverse deflection,
k_1 and k_2 are the first and second parameters of the two-parameter Pasternak foundation,
P is the compressive load,
E is the Young's modulus of elasticity of the beam,
I is the moment of inertia
$q(x)$ is the distributed transverse load.

For homogeneous prismatic thin beams free of distributed transverse load, the fourth order ordinary differential equation (ODE) becomes:

$$EI\frac{d^4v(x)}{dx^4} + P\frac{d^2v(x)}{dx^2} + k_1v(x) - k_2\frac{d^2v(x)}{dx^2} = 0 \tag{2}$$

Dividing by EI and re-arranging the equation gives:

$$\frac{d^4v(x)}{dx^4} + \left(\frac{P}{EI} - \frac{k_2}{EI}\right)\frac{d^2v(x)}{dx^2} + \frac{k_1}{EI}v(x) = 0 \tag{3}$$

Let $$\alpha = \frac{P}{EI} \tag{4}$$

$$\beta_1 = \frac{k_1}{EI} \tag{4b}$$

$$\beta_2 = \frac{k_2}{EI} \tag{4c}$$

Then, we have:

$$\frac{d^4v(x)}{dx^4} + (\alpha - \beta_2)\frac{d^2v(x)}{dx^2} + \beta_1v(x) = 0 \tag{5}$$

3. Research Methodology

The Stodola-Vianello iteration equations are found using successive integrations as follows:

$$\frac{d^4v(x)}{dx^4} = -(\alpha - \beta_2)\frac{d^2v(x)}{dx^2} - \beta_1v(x) \tag{6}$$

Integrating,

$$\int_0^x \frac{d^4v(x)}{dx^4}dx = -\int_0^x \left((\alpha - \beta_2)\frac{d^2v(x)}{dx^2} + \beta_1v(x)\right)dx \tag{7}$$

Hence,

$$\frac{d^3v(x)}{dx^3} = -(\alpha - \beta_2)\frac{dv(x)}{dx} - \beta_1\int_0^x v(x) + c_1 \tag{8}$$

where c_1 is an integration constant.
integrating again,

$$\frac{d^2v(x)}{dx^2} = -(\alpha - \beta_2)v(x) - \beta_1\int_0^x\int_0^x v(x)dxdx + c_1x + c_2 \tag{9}$$

where c_2 is the second integration constant.
Integrating again,

$$\theta(x) = \frac{dv(x)}{dx} = -(\alpha - \beta_2)\int_0^x v(x)\,dx - \beta_1\int_0^x\int_0^x\int_0^x v(x)\,dxdxdx + \frac{c_1x^2}{2} + c_2x + c_3 \tag{10}$$

where c_3 is the third constant of integration.
Integrating Equation (10),

$$v(x) = -(\alpha - \beta_2)\int_0^x\int_0^x v(x)\,dxdx - \beta_1\int_0^x\int_0^x\int_0^x\int_0^x v(x)\,dxdxdxdx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (11)$$

where c_4 is the fourth integration constant.

The four integration constants are found by enforcing the boundary conditions of the problem. Hence the Stodola-Vianello iteration becomes for the $(n + 1)$ iteration.

$$v_{n+1}(x) = -(\alpha - \beta_2)\int_0^x\int_0^x v_n(x)\,dxdx - \beta_1\int_0^x\int_0^x\int_0^x\int_0^x v_n(x)\,dxdxdxdx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (12)$$

4. Results and Discussion

The thin beam on Pasternak foundation problem shown in Figure 1 is studied. The ends $x = 0$, and $x = l$ are simply supported and l is the length of the beam.

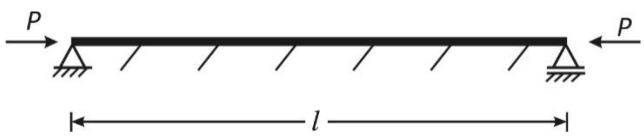

Figure 1: Pinned-pinned beam on Pasternak foundation under compressive load

The boundary conditions are given by Equations (13a) – (13d):

$$v(x = 0) = 0 \quad (13a)$$

$$v''(x = 0) = 0 \quad (13b)$$

$$v(x = l) = 0 \quad (13c)$$

$$v''(x = l) = 0 \quad (13d)$$

Enforcing the boundary conditions Equations (13a) and (13b) gives:

$$c_4 = 0 \quad (14a)$$

$$c_2 = 0 \quad (14b)$$

Hence, Equation (12) simplifies as:

$$v_{n+1}(x) = -(\alpha - \beta_2)\int_0^x\int_0^x v_n(x)\,dxdx - \beta_1\int_0^x\int_0^x\int_0^x\int_0^x v_n(x)\,dxdxdxdx + \frac{c_1x^3}{6} + c_3x \quad (15)$$

The nth buckling mode shape function that satisfies the boundary conditions is

$$v_n(x) = \sin\frac{n\pi x}{l} \quad (16)$$

Substituting Equation (16) into Equation (15) gives:

$$v_{n+1}(x) = -(\alpha - \beta_2)\int_0^x\int_0^x \sin\frac{n\pi x}{l}\,dxdx - \beta_1\int_0^x\int_0^x\int_0^x\int_0^x \sin\frac{n\pi x}{l}\,dxdxdxdx + \frac{c_1x^3}{6} + c_2x \quad (17)$$

Simplifying,

$$v_{n+1}(x) = (\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \beta_1\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi x}{l} + \frac{c_1x^3}{6} + c_3x \quad (18)$$

From Equation (9) after substituting Equation (10) and simplifying,

$$v''_{n+1}(x) = -(\alpha - \beta_2)\sin\frac{n\pi x}{l} - \beta_1\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + c_1 x \quad (19)$$

Enforcing the boundary conditions at $x = l$, gives:

$$v''_{n+1}(x = l) = -(\alpha - \beta_2)\sin n\pi - \beta_1\left(\frac{l}{n\pi}\right)^2 \sin n\pi + c_1 l = 0 \quad (20)$$

$$\therefore \qquad c_1 = 0 \quad (21)$$

Similarly, enforcing boundary conditions in Equation (17) gives:

$$v_{n+1}(x = l) = (\alpha - \beta_2)\sin n\pi - \beta_1\left(\frac{l}{n\pi}\right)^4 \sin n\pi + \frac{c_1 l^3}{6} + c_3 l = 0 \quad (22)$$

$$\therefore \qquad c_3 = 0 \quad (23)$$

Hence, the Stodola-Vianello iteration equation for the studied problem simplifies to:

$$v_{n+1}(x) = (\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \beta_1\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi x}{l} \quad (24)$$

Simplifying gives:

$$v_{n+1}(x) = \left((\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 - \beta_1\left(\frac{l}{n\pi}\right)^4\right)\sin\frac{n\pi x}{l} \quad (25)$$

Further simplification yields:

$$v_{n+1}(x) = \left((\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 - \beta_1\left(\frac{l}{n\pi}\right)^4\right)v_n(x) \quad (26)$$

At convergence,

$$v_{n+1}(x) = v_n(x) \quad (27)$$

$$\text{or } v_{n+1}(x) - v_n(x) = 0 \quad (27a)$$

Hence the characteristic equation is:

$$\left((\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 - \beta_1\left(\frac{l}{n\pi}\right)^4 - 1\right)v_n(x) = 0, \text{ where, } v_n(x) \neq 0$$

For nontrivial solutions,

$$(\alpha - \beta_2)\left(\frac{l}{n\pi}\right)^2 - \beta_1\left(\frac{l}{n\pi}\right)^4 = 1 \quad (28)$$

Re-arranging, we obtain:

$$(\alpha - \beta_2) = \left(1 + \beta_1\left(\frac{l}{n\pi}\right)^4\right)\left(\frac{n\pi}{l}\right)^2 \quad (29)$$

Thus,

$$\alpha = \beta_2 + \left(1 + \beta_1\left(\frac{l}{n\pi}\right)^4\right)\left(\frac{n\pi}{l}\right)^2 = \frac{P}{EI} \quad (30)$$

Then, the buckling load at the *n*th buckling mode is:

$$P_n = EI\left(\beta_2 + \left(1 + \beta_1\left(\frac{l}{n\pi}\right)^4\right)\left(\frac{n\pi}{l}\right)^2\right) \quad (31)$$

Simplification yields:

$$P_n = \frac{EI}{l^2}\left(\beta_2 l^2 + l^2\left(1 + \beta_1\left(\frac{l}{n\pi}\right)^4\right)\left(\frac{n\pi}{l}\right)^2\right) \quad (32)$$

Further simplification gives the standard form:

$$P_n = \frac{EI}{l^2}\left(\beta_2 l^2 + (n\pi)^2\left(1+\beta_1\left(\frac{l}{n\pi}\right)^4\right)\right) \quad (33)$$

When the second Pasternak foundation parameter vanishes, $\beta_2 = 0$ the foundation becomes a Winkler foundation and Equation (33) simplifies to:

$$P_n = \frac{EI}{l^2}\left((n\pi)^2\left(1+\beta_1\left(\frac{l}{n\pi}\right)^4\right)\right) \quad (33a)$$

Expressing Equation (33) in the standard form by defining the buckling load coefficient gives:

$$P_n = \frac{EI}{l^2}K(\beta_1, \beta_2, n) \quad (34)$$

where,

$$K(\beta_1, \beta_2, n) = \beta_2 l^2 + (n\pi)^2\left(1+\beta_1\left(\frac{l}{n\pi}\right)^4\right) \quad (35)$$

$K(\beta_1, \beta_2, n)$ is the buckling load coefficient for the *n*th buckling mode.

It is noted that if $\beta_2 = 0$,

$$K(\beta_1, \beta_2 = 0, n) = (n\pi)^2\left(1+\beta_1\left(\frac{l}{n\pi}\right)^4\right) \quad (35a)$$

The critical buckling load (P_{cr}) is found as the least buckling load and this occurs at the first buckling mode when $n = 1$.

Thus,

$$P_{cr} = P_{(n=1)} = \frac{EI}{l^2}\left(\beta_2 l^2 + \pi^2\left(1+\beta_1\left(\frac{l}{\pi}\right)^4\right)\right) \quad (36)$$

Hence,

$$P_{cr} = \frac{EI}{l^2}K(\beta_1, \beta_2, n=1) = \frac{EI}{l^2}K_{cr} \quad (37)$$

where

$$K(\beta_1, \beta_2, n=1) = K_{cr} = \beta_2 l^2 + \pi^2\left(1+\beta_1\left(\frac{l}{\pi}\right)^4\right) \quad (38)$$

Let $\lambda^2 = K(\beta_1, \beta_2, n=1) = K_{cr}$ (39)

Then,

$$K_{cr} = \frac{P_{cr} l^3}{EI} = \lambda^2 \quad (40)$$

$$\lambda = \sqrt{\frac{P_{cr} l^2}{EI}} \quad (41)$$

Table 1: Values of critical buckling load parameters for various values of the Pasternak foundation parameters

		$\bar{k}_2 = 0 = \beta_2 (l/\pi)^2$		% Difference	
$\bar{k}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Present	Present andTaha (2014)	Present andAnghel and Mares (2019)
0	3.1415	3.1413	3.141593	0.00296	0.0093
100	4.4723	4.4721	4.472329	0.000648	0.0051
		$k_2 = 1 = \beta_2 (l/\pi)^2$			
$\bar{k}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Present		
0	4.4428	4.4427	4.442883	0.00187	0.00412
100	5.4654	5.4653	5.465467	0.000123	0.0031
		$\bar{k}_2 = 2.5 = \beta_2 (l/\pi)^2$			
$\bar{k}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Present		
0	5.8774	5.8772	5.877382	–0.00031	0.0031
100	6.6840	6.6838	6.683991	–0.000135	0.00286

The boundary value problem of thin beam resting on two-parameter Pasternak foundation has been solved in this paper using Stodola-Vianello iteration method. The problem is represented for homogeneous prismatic beams by a fourth order ODE subject to boundary conditions determined from the end support conditions. The Stodola-Vianello iteration equation was developed using the method of four successive integrations after re-arranging the ODE for the case of Dirichlet boundary conditions. A trigonometric shape function for the *n*th buckling mode which satisfies the simply supported end conditions is used in the developed Stodola-Vianello iteration formula to find the $(n+1)$th buckling mode function. The Dirichlet boundary conditions are used to obtain the integration constants. The condition for convergence of the iteration is used to obtain the characteristic buckling equation as Equation (28).

The root of the characteristic equation gives the eigenvalue, α as Equation (30). The *n*th buckling load (P_n) is found as Equation (31), and in standard form as Equation (33), where the *n*th buckling load coefficient is found as Equation (35). The expression for the *n*th buckling load which is presented in Equation (33) for BoPF reduces to the expression for the *n*th buckling load for a beam on Winkler foundation BoWF when the second Pasternak foundation parameter k_2 vanishes. Similarly, the expression for critical buckling load P_{cr} for BoPF simplifies to the expression for P_{cr} for BoWF when the second Pasternak foundation parameter, k_2, vanishes. The critical buckling load P_{cr} is the least buckling load and occurs at the first buckling mode. P_{cr} is found as Equation (36) from Equation (35). The critical buckling load coefficient $K(\beta_1\beta_2, n=1) = K_{cr}$ is found as Equation (38).

Values of the critical load buckling parameter λ which is defined in terms of K_{cr} and P_{cr}, l, EI in Equations (40) and (41) are presented in Table 1 for various values of β_1 and β_2 and compared with previous results obtained by Taha (2014) and Anghel and Mares (2019). Table 1 illustrates the close agreement between the present results and the results by Taha (2014) and Anghel and Mares (2019). The percentage difference between present exact critical values of K_{cr} and hence P_{cr} and previous results by Taha (2014) and Anghel and Mares (2019) are less than 0.004% for all values of $0 \le \bar{k}_1 \le 100,$ and $0 \le \bar{k}_2 \le 2.5.$ The closed form analytical expression for the buckling loads of all the modes of buckling of the simply supported thin BoPF has been derived in this study using Stodola-Vianello iteration method. The exact buckling load solutions developed was due to the exact shape function for the thin beam with Dirichlet boundary conditions used in the Stodola-Vianello iteration.

5. Conclusions

In conclusion, this paper has studied the Stodola-Vianello iteration method for the buckling load analysis of Euler-Bernoulli BoPF. The governing ODE for BoPF was reformulated using four successive integrations to a Stodola-Vianello iteration equation with four integration constants, corresponding to the fourth order of the governing ODE. The four constants of integration were determined using the Dirichlet boundary conditions of the BoPF. The BoPF problem consequently becomes simplified to a more readily solved algebraic iteration problem.

(i) Stodola-Vianello iteration method simplifies the buckling load problem of a thin beam on Pasternak foundation to an iterative equation for deriving the $(n + 1)$th buckling mode shape function using the nth buckling eigenfunction.

(ii) The iteration equation contains four constants of integration which are found using the end support conditions of the problem.

(iii) The buckling mode shape function is found from the boundary conditions.

(iv) The requirement for convergence of the iteration is used to find the characteristic buckling equation.

(v) The roots of the characteristic buckling equation are used to find the buckling load expression.

(vi) The minimum buckling load occurs at the first buckling mode and gives the critical buckling load.

(vii) The critical buckling load (P_{cr}) expression is the exact expression because the exact buckling shape function was used and the governing ODE was satisfied at all points on the solution domain.

(viii) Results obtained for the critical buckling load parameters agree with previous solutions by Taha (2014) and Anghel and Mares (2019).

(ix) Expectedly the nth buckling load P_n solutions obtained in this study simplified to the P_n solutions for BoWF when the second Pasternak foundation parameter, k_2, (or β_2) vanishes.

Nomenclature/Symbols/Abbreviations

EBBT	Euler-Bernoulli Beam Theory
TBT	Timoshenko Beam Theory
ODE(s)	Ordinary Differential Equation(s)
VIM	Variational Iteration Method
RDM	Recursive Differentiation Method
BVP	Boundary Value Problem
BoWF	Beam on Winkler Foundation
BoPF	Beam on Pasternak Foundation
x	longitudinal coordinate axis of the beam
$v(x)$	transverse deflection
k_1	first parameter of the two-parameter Pasternak foundation
k_2	second parameter of the two-parameter Pasternak foundation
P	compressive load on beam on Pasternak foundation
E	Young's modulus of elasticity of the beam
I	moment of inertia
$q(x)$	distributed transverse load
α	parameter defined in terms of P and EI
β_1	parameter defined in terms of k_1 and EI
β_2	parameter defined in terms of k_2 and EI
c_1, c_2, c_3, c_4	integration constants
$\theta(x)$	slope of beam
n	buckling mode
l	length of beam
$v_n(x)$	nth buckling mode shape function

P_n — nth buckling load
P_{cr} — critical buckling load
$K(\beta_1, \beta_1, n)$ — buckling load coefficient for the beam on Pasternak foundation for the nth buckling mode
λ — critical buckling load parameter defined in terms of P_{cr}, l and EI
$\bar{k}_1$ — dimensionless foundation parameter defined in terms of β_1 and l
$\bar{k}_2$ — dimensionless foundation parameter defined in terms of β_2 and (l/π)

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