UJEAS

UNIZIK JOURNAL OF ENGINEERING AND APPLIED SCIENCES

UNIZIK Journal of Engineering and Applied Sciences 2(1), June 2023, 250-259
Journal homepage: https://journals.unizik.edu.ng/index.php/ujeas
PRINT ISSN: 2992-4383 || ONLINE ISSN: 2992-4391

Stodola-Vianello method for the buckling load analysis of Euler-Bernoulli beam on Winkler foundation

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Abstract
The critical buckling load determination for Euler-Bernoulli (EB) beams on elastic foundations is an important consideration in their analysis especially when they are subjected to in-plane compressive loads. This paper presents the Stodola-Vianello method for the iterative presentation and solution of the governing fourth order ordinary differential equation (ODE). The method of successive integrations is used after re-arrangement to express the governing ODE in iterative form. The eigenfunction for the *n*th buckling mode is used to obtain the $(n + 1)$th buckling mode equation for the Dirichlet end conditions considered. The condition for convergence at the *n*th buckling mode is used to drive the eigenequation, whose roots were used to find the buckling loads at the *n*th mode. The critical buckling mode is exact since the exact buckling mode function was used to derive it. The critical buckling load expression was expressed in standard form in terms of critical buckling load coefficients which depend on the parameter representing the EB beam on Winkler foundation interaction. The critical buckling load coefficients were found to be in close agreement with previously reported works, and identical with the exact critical buckling load coefficient obtained by previous researchers using closed form analytical methods.

Keywords: Stodola-Vianello iteration method, eigenfunction, eigenvalue, critical buckling load coefficient, beam on Winkler foundation.

1. Introduction

The analysis and design of buried structures is a soil-structure interaction problem. The soil supports the buried pipelines and alters the equation of equilibrium. It is thus important to evaluate the soil reaction or loading on buried structures in order to adequately analyse the structural behaviour. In geotechnical engineering, some buried structural elements like pipelines, shallow foundations, and piles can be idealized as beam structures and the resulting soil-structure interaction problem called beam on elastic foundation due to the idealization of the surrounding soil, as elastic foundation.

There are two common types of beam theories, namely: Euler-Bernouli beam theory (EBBT) and Timoshenko beam theory (TBT). Other theories for beams include Vlasov beam theory, Mindlin beam theory, and shear deformation beam theories developed by Levinson (1981), Dahake and Ghugal (2013) and Sayyad and Ghugal (2011). Shimpi. Elastic foundation models that have been used are Winkler, Pasternak, Vlasov, Hetenyi, Kerr. EBBT which is suitable for thin beams, for which the thickness, h, to span, l, ratio is less than 0.05, ($h/l \leq 0.05$), ignores shear deformation and assumes that planes orthogonal to the longitudinal fibres remain plane and orthogonal after deformation. The assumptions make the EBBT ideal for thin beams for which shear deformation do not contribute significantly to their behaviour (Ike, 2018a). For moderately short and deep beams, TBT is used to account for the effect of shear deformation.

Winkler model which is the simplest elastic foundation model assumes the soil as a series of closely spaced, mutually independent, linear elastic vertical springs as shown in Figure 1, which provide stiffness that is directly proportional to the deflection of the beam at the point.

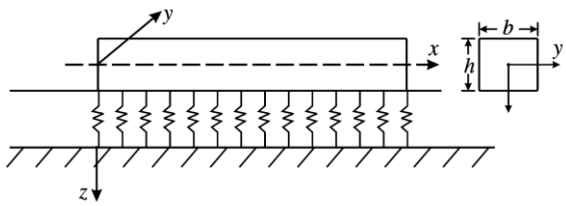

Figure 1: Typical thin beam of rectangular cross-section on Winkler model of soil idealized as a bed of closely spaced vertical elastic springs which do not interact with neighbouring springs

Hence the soil reaction is defined using a single parameter k called the Winkler parameter which represents the soil stiffness parameter. The major limitation of the Winkler model is the lack of continuity of the soil spring model, which results in the inability of the model to consider the shear interaction between adjacent springs.

Several two-parameter foundation models which consider the interaction between adjoining springs have been derived by Pasternak, Vlasov, Hetenyi and others. Two-parameter elastic foundation models use two foundation parameters to describe the soil reaction; the first parameter k_1 represents the vertical spring stiffness, as in the Winkler model, while the second parameter k_2 represents the coupling effect of the linear elastic springs.

Beams on elastic foundations which are subjected to axial compressive load are prone to buckling failures when such axial compressive loads attain certain critical values. The determination of the critical buckling loads become main issues for their analysis and design.

Hetenyi (1946) was a pioneer in studying in the buckling of beams on elastic foundations; with his use of trial function methods to calculate critical buckling loads of beams on Winkler foundations. Timoshenko and Gere (1985) have presented closed form solutions for critical buckling loads for uniform simply supported thin beams on elastic foundations. Taha and Hadima (2015) used the recursive differentiation method (RDM) to obtain mathematical solutions for critical buckling loads of non-uniform beams on elastic foundations.

Atay and Coskun (2009) used the variational iteration method (VIM) to solve the stability of beams on elastic foundation problems. Taha (2014) applied the recursive differentiation method to the analysis of a beam-column on an elastic foundation. Anghel and Mares (2019) used collocation method based on integral formulation for the stability problem to develop accurate solutions to the beam on elastic foundation problem. Aristizabal-Ochoa (2013) developed solutions to the buckling of slender columns on an elastic foundation with generalized end conditions. Hassan (2018) presented solutions to the buckling problems of beams on elastic foundations for different boundary conditions.

Soltani (2020) used the finite element method to develop buckling solutions for variable cross-section axially functionally graded Timoshenko beam on elastic foundation. Ike (2018b) used the finite sine transformation method to find the exact solutions to the eigenvalue problem of free vibrating thin beam on Winkler foundation. Ofondu et al (2018) used the Stodola-Vianello iteration method to derive accurate critical buckling load solutions for Euler columns. They derived the Stodola-Vianello iteration formula for the Euler column buckling problem; and used an algebraic basis function for the clamped-pinned end conditions considered to derive successive iterates for the basis buckling function. They found that a few iterative steps gave accurate critical buckling load results.

Wang et al (2005) have provided exact solutions for the buckling of beams on elastic foundations for various boundary conditions. Huang and Luo (2011) solved the buckling problem of beams on elastic foundation by expanding the mode shape as power series, thus transforming the field equations into algebraic equations. Critical buckling loads are then found by them by using the conditions for existence of nontrivial solutions. Literature reveals that the Stodola-Vianello iteration method has not beam used to formulate and solve stability problems of BoWF. The formulation and solution of the BoWF is the main focus of this work.

In this paper the Stodola-Vianello iterative method is applied to develop buckling load solutions for thin beam on Winkler foundations with pinned-pinned ends. The innovation in the paper is the first principles approach to the Stodola-Vianello iteration formulation of the governing equation of the BoWF. Another innovative aspect is the first principles systematic use of exact shape functions of Dirichlet simply supported beams to find exact buckling load solutions.

2. Governing Equation/Theory

The equation for the buckling load problem of a thin beam on Winkler foundation is given by

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w(x)}{dx^2}\right)+P\,\frac{d^2w(x)}{dx^2}+kw(x)=p(x) \qquad (1)$$

where $w(x)$ is the transverse deflection,
x is the longitudinal coordinate axis of the beam,
$p(x)$ is the distributed transverse load on the span
k is the Winkler modulus of the foundation,
P is the compressive load,
E is the Young's modulus of elasticity of the beam,
I is the moment of inertia.

In the absence of distributed transverse load $p(x)=0$, and for. homogeneous prismatic beams, Equation (1) simplifies to the fourth order ordinary differential equation:

$$EI\,\frac{d^4w}{dx^4}+P\,\frac{d^2w}{dx^2}+kw=0 \qquad (2)$$

Dividing by EI, yields:

$$\frac{d^4w}{dx^4}+\frac{P}{EI}\frac{d^2w}{dx^2}+\frac{k}{EI}w=0 \qquad (3)$$

Let $\alpha=\frac{P}{EI}$ (4a)

$$\beta=\frac{k}{EI} \qquad (4b)$$

Then the equation s:

$$\frac{d^4 w}{dx^4} + \alpha \frac{d^2 w}{dx^2} + \beta w = 0 \quad (5)$$

3. Methodology

The Stodola-Vianello iteration equations are derived by successive integrations of the governing equation after re-arrangement. Thus

$$\frac{d^4 w}{dx^4} = -\left(\alpha \frac{d^2 w}{dx^2} + \beta w \right) \quad (6)$$

Integrating once,

$$\int_0^x \frac{d^4 w}{dx^4} dx = -\int_0^x \left(\alpha \frac{d^2 w}{dx^2} + \beta w \right) dx \quad (7)$$

Hence,

$$\frac{d^3 w}{dx^3} = -\alpha \frac{dw}{dx} - \beta \int_0^x w\, dx + c_1 \quad (8)$$

where c_1 is an integration constant.
Integrating again,

$$\frac{d^2 w}{dx^2} = -\alpha w(x) - \beta \int_0^x \int_0^x w(x) dx dx + c_1 x + c_2 \quad (9)$$

where c_2 is the second integration constant.
Integrating again,

$$\theta(x) = \frac{dw}{dx} = -\alpha \int_0^x w(x) dx - \beta \int_0^x \int_0^x \int_0^x w(x) dx dx dx + \frac{c_1 x^2}{2} + c_2 x + c_3 \quad (10)$$

where c_3 is an integration constant.
Integrating Equation (10),

$$w(x) = -\alpha \int_0^x \int_0^x w(x) dx dx - \beta \int_0^x \int_0^x \int_0^x \int_0^x w(x) dx dx dx dx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \quad (11)$$

where c_4 is the fourth integration constant.

The four integration constants are determined using the boundary conditions of the problem. Hence the Stodolla-Vianello iteration becomes for the (n + 1)th iteration:

$$w_{n+1}(x) = -\alpha \int_0^x \int_0^x w_n(x) dx dx - \beta \int_0^x \int_0^x \int_0^x \int_0^x w_n(x) dx dx dx dx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \quad (12)$$

4. Results and Discussion

The pinned-pinned beam on Winkler foundation shown in Figure 2 is considered.

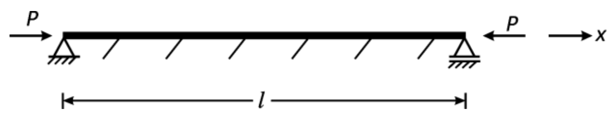

Figure 2: Pinned-pinned beam on Winkler foundation under compressive load

For beam of length *l*,
The boundary conditions are given by the four equations:

$$w(x=0)=0 \tag{13a}$$

$M(x=0)=0,$ where $M(x=0)$ is the bending moment distribution at $x=0.$
Using bending moment deflection equation, the left support,

$$\frac{d^2w}{dx^2}(x=0)=0 \tag{13b}$$

$$w(x=l)=0 \tag{13c}$$

$M(x=l)=0,$ where $M(x)$ is the bending moment distribution.
Using bending moment deflection equation, at the right support,

$$\frac{d^2w}{dx^2}(x=l)=0 \tag{13d}$$

Substituting the boundary conditions at $x = 0$ in the Stodola-Vianello equations give:

$$c_4 = 0 \tag{14a}$$

$$c_2 = 0 \tag{14b}$$

Hence, Equation (12) is simplified after considering boundary conditions at the left support $(x=0)$ to the following iteration equation:

$$w_{n+1}(x) = -\alpha\int_0^x\int_0^x w_n(x)\,dxdx - \beta\int_0^x\int_0^x\int_0^x\int_0^x w_n(x)\,dxdxdxdx + \frac{c_1x^3}{6} + c_3x \tag{15}$$

A buckling shape function at the *n*th mode that satisfies the pinned-pinned boundary conditions at the left and right supports is the sinusoidal function:

$$w_n(x) = \sin\frac{n\pi x}{l} \tag{16}$$

Then substituting in Equation (15) gives:

$$w_{n+1}(x) = -\alpha\int_0^x\int_0^x \sin\frac{n\pi x}{l}\,dxdx - \beta_1\int_0^x\int_0^x\int_0^x\int_0^x \sin\frac{n\pi x}{l}\,dxdxdxdx + \frac{c_1x^3}{6} + c_2x \tag{17}$$

Evaluating the integrals in Equation (17) gives a simplified iteration as:

$$w_{n+1}(x) = \alpha\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \beta_1\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi x}{l} + \frac{c_1 x^3}{3} + c_3 x \quad (18)$$

The Stodola-Vianello iteration constructed using Equation (9) gives after evaluation of the integral:

$$w''_{n+1}(x) = -\alpha \sin\frac{n\pi x}{l} - \beta\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + c_1 x \quad (19)$$

Enforcing the boundary conditions at $x = l$, give:

$$w''_{n+1}(x = l) = -\alpha \sin n\pi - \beta\left(\frac{l}{n\pi}\right)^2 \sin n\pi + c_1 l = 0 \quad (20)$$

$$\therefore \quad c_1 = 0 \quad (21)$$

Similarly, applying the boundary conditions at $x = l$ for $w(x)$ in Equation (18) gives:

$$w_{n+1}(x = l) = \alpha\left(\frac{l}{n\pi}\right)\sin n\pi - \beta\left(\frac{l}{n\pi}\right)^4 \sin n\pi + \frac{c_1 l^3}{6} + c_3 l = 0 \quad (22)$$

$$\therefore \quad c_3 = 0 \quad (23)$$

Thus, Stodola-Vianello iteration formula for the pinned-pinned BoWF becomes:

$$w_{n+1}(x) = \alpha\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \beta\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi x}{l} \quad (24)$$

Simplifying,

$$w_{n+1}(x) = \left(\alpha\frac{l^2}{(n\pi)^2} - \beta\frac{l^4}{(n\pi)^4}\right)\sin\frac{n\pi x}{l} \quad (25)$$

Further simplification gives:

$$w_{n+1}(x) = \left(\alpha\frac{l^2}{(n\pi)^2} - \beta\frac{l^4}{(n\pi)^4}\right)w_n(x) \quad (26)$$

At convergence,

$$w_{n+1}(x) = w_n(x) \quad (27)$$

Hence the characteristic buckling equation becomes:

$$\alpha\left(\frac{l}{n\pi}\right)^2 - \beta\left(\frac{l}{n\pi}\right)^4 = 1 \quad (28)$$

Solving for α gives:

$$\alpha = \left(1 + \frac{\beta l^4}{(n\pi)^4}\right)\left(\frac{n\pi}{l}\right)^2 \tag{29}$$

Thus, the nth buckling load P_n is:

$$P_n = EI\left(\left(\frac{n\pi}{l}\right)^2 + \frac{\beta l^2}{(n\pi)^2}\right) \tag{30}$$

or,

$$P_n = \frac{EI}{l^2}\left((n\pi)^2 + \frac{\beta l^4}{(n\pi)^2}\right) \tag{30a}$$

The critical buckling load is the minimum value for P_n and corresponds to $n = 1$.
Thus,

$$P_{cr} = P_{(n=1)} = EI\left(\left(\frac{\pi}{l}\right)^2 + \frac{\beta l^2}{\pi^2}\right) \tag{31}$$

$$P_{cr} = \frac{EI}{l^2}\left(\pi^2 + \frac{\beta l^4}{\pi^2}\right) = K_{cr}\frac{EI}{l^2} \tag{32}$$

where $K_{cr}(\beta l^4) = \pi^2 + \dfrac{\beta l^4}{\pi^2}$ (33)

$K_{cr}(\beta l^4)$ is the critical buckling load coefficient

$K_{cr}(\beta l^4)$ is dependent upon βl^4 which depends on the Winkler parameter, k and the beam flexural rigidity EI as given by Equation (4b). $K_{cr}(\beta l^4)$ is calculated for various values of βl^4 and presented in Table 1 together with previous results obtained using VIM and exact methods.

Table 1: Values of critical buckling load coefficient $K_{cr}(\beta l^4)$ for various values of βl^4 for present work and previous research works

βl^4	Present work K_{cr}	(Atay and Coskun, 2009) VIM K_{cr}	(Wang et al, 2005) Exact K_{cr}
0	9.869604401	9.8696	9.869604401
50	14.93566358	14.9357	14.93566358
100	20.00172277	20.0017	20.00172277

Table 2 presents a comparative study of the present solution with the method of collocation presented by Anghel and Mares (2019).

Table 2: Comparison of present work with collocation results

βl^4	$n=10$	$n=20$	$n=40$	$n=60$	$n=100$	Present work
	Collocation method by Anghel and Mares (2019)			Anghel and Mares (2019)		
0	11.051	10.378	10.108	10.025	9.961	9.869604401
50	16.756	15.712	15.298	15.172	15.075	14.93566358
100	22.416	21.032	20.485	20.318	20.188	20.00172277

The Stodolla-Vianello method has been used in this paper to develop buckling load solutions for thin beam on Winkler foundation. The Stodolla-Vianello iteration equation was derived from first principles for beam on Winkler foundation with pinned ends as Equation (15).

A buckling shape function at the *n*th buckling mode that satisfies the boundary conditions at the pinned ends is chosen as the eigenfunction of a vibrating thin beam as Equation (16). The integration constants are obtained using the boundary conditions. The requirement of convergence as presented in Equation (27) is used to obtain the characteristic eigenequation as Equation (28), which is solved to find the eigenvalue α as Equation (29). The *n*th buckling load (P_n) is found as Equation (30). The exact mathematical expression for the buckling load for all the modes of buckling of the simply supported thin BoWF is thus obtained in this study. This is made possible by the use of exact shape function for the thin beam with Dirichlet boundary conditions employed in the Stodola-Vianello iteration

The critical buckling load P_{cr} corresponds to the minimum value for P_n which is found at the first buckling mode (n = 1). The critical buckling load is given by Equation (32) which is found to be the exact expression since the governing differential equation is satisfied at all points on the domain and the boundary conditions are also satisfied. Table 1 which shows the variation of the critical buckling load coefficient K_{cr} with the Winkler foundation parameter (βl^4) illustrates the identical results of the Stodola-Vianello iteration at the *n*th iteration and the exact results obtained by Timoshenko and Gere (1985) and Wang et al (2005). The present results are also in close agreement with variational iteration method (VIM) results presented by Atay and Coskun (2009). Table 2 shows that the present results also agree remarkably well with Anghel and Mares (2019) collocation results using $n=100$ collocation points. The results further show that an increase in the Winkler foundation parameter (βl^4) results in an increase in the critical buckling load P_{cr}.

5. Conclusion

In conclusion, this paper has presented Stodolo-Vianello iteration method for the buckling load analysis of thin BoWF. The governing equation BoWF was reformulated using four successive integrations to Stodola-Vianello iteration equation which contains four integration constants determined using the problem boundary conditions. The problem consequently becomes simplified to an easier to solve algebraic problem.

(i) Stodola-Vianello iteration method reduces the buckling load problem of a thin beam on Winkler foundation to an iterative equation used to derive the (n + 1)th buckling mode shape function from the *n*th buckling mode shape function.
(ii) The Stodola-Vianello iteration equation is derived such that the buckling mode shape function satisfies the boundary conditions.
(iii) The condition for convergence is used to derive the characteristic eigenequation.
(iv) The roots of the eigenequation are used to obtain the buckling load expression for the *n*th buckling mode.
(v) The least buckling load corresponds to the first buckling mode and yields the critical buckling load.
(vi) The critical buckling load expression obtained is the exact expression for P_{cr} since the exact buckling eigenfunction was used to derive the Stodola-Vianello iteration equation.

Nomenclature

EBBT	Euler-Bernoulli Beam Theory
TBT	Timoshenko beam theory
BoWF	Beam on Winkler Foundation
ODE	Ordinary Differential Equation
$w(x)$	transverse deflection
x	longitudinal coordinate axis of beam
y	coordinate axis of beam in the dimension of width
z	transverse coordinate axis
$p(x)$	distributed transverse load on the span
k	Winkler modulus of the Winkler foundation
P	compressive load
E	Young's modulus of elasticity of the beam
I	moment of inertia
α	parameter defined in terms of P and EI
β	parameter defined in terms of k and EI
$\frac{d}{dx}$	first derivative with respect to x
$\frac{d^2}{dx^2}$	second derivative with respect to x
$\int \ldots dx$	integration with respect to x
c_1, c_2, c_3, c_4	constants of integration
$\theta(x)$	slope of beam
$M(x)$	bending moment
l	beam length, span of beam
$\iint \ldots dx\ dx$	two successive integrals with respect to x
$\iiiint \ldots dxdxdxdx$	four successive integrals with respect to x
n	buckling mode number
b	width of beam
h	thickness of beam
P_n	nth buckling mode
K_{cr}	critical buckling load coefficient
P_{cr}	critical buckling load

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