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This special issue is dedicated to Professor Onukwuli Okechukwu Dominic (FAS), marking his retirement and celebrating a remarkable career. His legacy of exemplary scholarship, mentorship, and commitment to advancing knowledge is commemorated in this collection of works.

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A Hybrid Liu-Ridge Method of Handling Multicollinearity in Linear Regression Models

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Abstract
Multicollinearity is a critical challenge in linear regression analysis, causing instability and unreliability in Ordinary Least Squares (OLS) estimates when independent variables are highly correlated. While existing biased estimators, such as Ridge Regression, Liu Estimator, and Kibria-Lukman Estimator, partially address this issue, they often fall short of achieving optimal Mean Squared Error (MSE) performance across varying conditions. This study introduces the Hybrid Liu-Ridge (HLR) estimator, a novel integration of the Modified Ridge Type (MRT) and Modified Liu (MLIU) estimators, designed to robustly handle multicollinearity. A comprehensive theoretical analysis demonstrates the superiority of the HLR estimator, particularly in minimizing MSE compared to its predecessors. Performance evaluation via Monte Carlo simulations, conducted under varying multicollinearity levels, error variances, and sample sizes, confirms the consistency of the HLR estimator in outperforming existing methods. Real-world applications using agricultural and economic data further validate its robustness and practical utility. By offering improved reliability and adaptability, the HLR estimator represents a significant advancement in addressing multicollinearity, providing researchers and practitioners with a superior tool for regression analysis.

Keywords: Multicollinearity, linear regression, Ordinary Least Squares, simulation, Monte Carlo experiment.

1.0 Introduction

The matrix form of linear regression model is as in equation (1)

$$y = X\beta + e\_i \tag{1}$$

Such that y is an $(n \times 1)$ vector dependent variable, X is a complete design matrix of $(n \times p)$ exogenous variables, β is an unknown parameter with $(n \times 1)$ vector, and e_i is an $(n \times 1)$ random error with variance$V(e_i) = \sigma^2 I_n$, whereby $\sigma^2$and I_n are unknown parameters and identity matrix of order n accordingly. The Ordinary Least Squares Estimator (OLSE), represented by equation (2), defined as follows:

$$\hat{\beta}_{OLS} = \Pi^{-1} X'Y \tag{2}$$

Where $\Pi = X'X$.

In the absence of multicollinearity among the explanatory variables, the Ordinary Least Squares (OLS) estimator is the Best Linear unbiased Estimator (BLUE), which is well recognized for being ideal in the context of traditional linear regression. But according to Swindel (1976), the OLS estimator performs badly, generating inaccurate projections and inconsistent estimates. The modified Jackknife Ridge Regression Estimator (Singh and Pandey, 2005), Liu estimator by Liu (1993), Ridge-Liu estimator (Gao and Liu, 2020), Inverse Gaussian Liu-Type Estimator (Akran *et al*. 2020), Bayesian ridge regression (Li *et al*., 2020), Bayesian elastic net regression (Liu *et al*., 2020), Neural network-based regression models (Huang *et al*., 2020). An unbiased Estimator with prior information (Lukman et al. 2020), and others are some of the biased estimators that have been developed to address this type of problem. Researchers have recently introduced two-parameter estimators, including the modified two-parameter, the modified ridge type (MRT) estimator (Lukman *et al.* 2019), the modified new two-type parameter estimator (Ahmad and Aslam, 2020), the new modified Liu Ridge-Type estimator (Oladapo *et al*. 2022), the new two-parameter ridge estimator (Owolabi *et al.* 2022), and the new two-parameter Kibria-Lukman (NTPKL) estimator (Idowu *et al.* 2023) are all used to counteract multicollinearity. The ordinary ridge regression estimator (ORRE), one of the most popular biased estimators, adds a positive biasing parameter, *k*, to the matrix's diagonal members in an effort to reduce multicollinearity. Nevertheless, choosing *k* is still quite difficult because it is essential to reducing the bias in the regression. The Liu estimator, proposed by Liu (1993), offers an advantage in selecting the biasing parameter *d*, but the ORRE's reliance on a nonlinear function of *k* makes it difficult to choose an optimal value. This study aims to develop a new estimator that can circumvent the problem of high correlation among the independent variables. The proposed estimator's performance is evaluated in comparison to several existing estimators, including the OLS estimator, Modified Liu estimator (MLIU), Ridge Regression (RR) estimator, among others. This study contributes to the ongoing discourse by proposing a novel biased estimator, termed Hybrid Liu-Ridge (HLR) estimator. The HLR estimator synthesizes the strengths of the Modified Ridge Type (MRT) and Modified Liu (MLIU) estimators, aiming to enhance MSE performance while maintaining robustness across varying levels of multicollinearity, error variances, and sample sizes. The theoretical properties of the HLR estimator are rigorously derived and validated through Monte Carlo simulations and real-world applications. By demonstrating consistent superiority in minimizing MSE compared to existing methods, the HLR estimator represents a significant advancement in regression analysis under multicollinearity, offering a more reliable and practical alternative for statistical modeling.

2.0 Materials and methods

2.1 Some already existing Shrinkage methods

2.1.1 Ridge Estimator.

To address the issue of multicollinearity in regression analysis, Heoerl and Kennard (1970) created the Ridge regression estimator. To lessen the collinearity effect, the Ridge parameter was included in the matrix. It is common for the OLS estimator to have a high variance, which is described as:

$$\hat{\beta}_{RE}(k) = \left(\Pi + kI_p\right)^{-1} X'Y\hat{\beta}_{OLS} \quad (3)$$

Where I is the (p x p) identity matrix and $k = \frac{p\sigma^2}{\sum_{i=1}^{p}\hat{\beta}_i}$ is the ridge parameter k. $\hat{\beta}$ is an unbiased estimator of $\hat{\beta}_k$, σ^2 is the Mean Square Error (MSE), and p is the number of explanatory variables.

2.1.2 Liu Estimator

Liu (1993) developed the Liu Estimator by fusing the ridge estimator with the Stein estimator to address the issue of multicollinearity in the data sets.

$$\hat{\beta}_L(d) = \left(\Pi + I_p\right)^{-1} \left(\Pi + dI_p\right)\hat{\beta}_{OLS} \quad (4)$$

Where $k = 1 - \sigma^2 \left[\frac{\sum_{i=1}^{p} \frac{1}{\lambda_1(\lambda_1 + 1)}}{\sum_{i=1}^{p} \frac{\alpha^2}{(\lambda_1 + 1)^2}}\right]$ such that λ_1 is the i^th^ Eigen value of Π

2.1.3 KL Estimator

Kibria and Lukman (2020) suggested an additional one-parameter estimator in addition to the ones that are currently in use to address the issue of multicollinearity in linear regression models. The estimator was defined as follows:

$$\hat{\beta}_{KL}(k) = \left(\Pi + kI_p\right)^{-1} \left(\Pi - kI_p\right)\hat{\beta}_{OLS} \quad (5)$$

Where $k = \frac{\hat{\sigma}^2}{2\hat{\sigma}^2 + \left(\hat{\sigma}^2 / \lambda_1\right)}$ and $\hat{\beta}_{OLS}$ is the estimator of Ordinary Least Square (OLS).

2.1.4 Two-parameter Liu-Ridge Estimator

Two-parameter Ridge-Liu estimator was introduced by Ozkale and kaciranlar in (2007) defined as:

$$\hat{\beta}_{TP}(k,d) = \left(\Pi + kI_p\right)^{-1} \left(X'Y + kd\hat{\beta}_{OLS}\right) \quad (6)$$

where I is an identity matrix, k is the Ridge parameter, d is the biasing parameter for Liu which both take value from 0 to 1 and $\hat{\beta}_{OLS}$is the least square estimator. Meanwhile, the introduction of TPE is to cater for the problem of multicollinearity in regression model.

2.1.5 MRT Estimator

The MRT estimator, which is described as follows, was suggested by Lukman *et al*. (2019) as an alternate estimator to address the multicollinearity issue in linear regression models.

$$\hat{\beta}_{MRT}(k,d) = \left(\Pi + k(I+d)\right)^{-1} X'Y \tag{7}$$

Where $k_{HMP}^{MRT} = \dfrac{p\sigma^2}{\sum_{i=1}^{p}(1+d)\alpha_i^2}$ and $d_{MRT} = \dfrac{p}{\sum_{i=1}^{p}\frac{1}{d}}$,k>0 and 0 <d < 1. They confirmed that, primarily when k=0 and d=0, the estimator can produce results that are equivalent to those of the OLS and the ridge estimator.

2.1.6 New Biased-Based Estimator

A new biased-based estimator was formulated by Sakallioglu and Kaciranlar (2008) and is described as follows:

$$\hat{\beta}_{NBB}(k,d) = (\Pi + I)^{-1}(\Pi + (k+d)I)\hat{\beta}_{ORR} \tag{8}$$

Where $\hat{\beta}_{ORR} = (\Pi + kI)^{-1} X'y$, k and d are the biasing parameters.

2.1.7 DK Estimator

To mitigate the effects of multicollinearity in linear regression models, Dawoud and Kibria (2020) introduced a novel biased estimator, which they defined as follows:

$$\hat{\beta}_{DK}(k,d) = (\Pi + k(1+d)I)^{-1}(\Pi - k(1+d)I)\hat{\beta}_{OLS} \tag{9}$$

2.2 The proposed estimator's theoretical methodology

In order to create Hybrid Liu-Ridge estimator, MRT estimator was integrated into MLIU estimator. The MLIU estimator is defined as follows:

$$\hat{\beta}_{ML}(d) = (\Pi + I_p)^{-1}(\Pi - dI_p)\hat{\beta}_{OLS} \tag{10}$$

MRT estimator as an alternative estimator to address the multicollinearity problem in linear regression models as given in equation (7), where $k> 0$ *and* $0 <d < 1$. Consequently, to explore additional biased estimators capable of effectively managing high multicollinearity in linear regression models, a new estimator, termed the Hybrid Liu-Ridge Estimator, is proposed and is defined as follows:

$$\hat{\beta}_{(k,d)}^{HLR} = (\Pi + I_p)^{-1}(\Pi - dI_p)(\Pi + k(1+d)I_p)^{-1}\Pi\hat{\beta}_{OLS} \tag{11}$$

The general linear regression model described in equation (1) can be expressed in the canonical form as following:

$$y = T\alpha + e_i \tag{12}$$

Where$T = XH$, $\alpha = H'\beta$ and the eigenvectors of Π are represented by the columns of the orthogonal matrix H. Then, $T'T = H'X'XH = \Pi = diag(\lambda_1,\ldots,\lambda_p)$, the ordered eigenvalues of Π are denoted by $\lambda_1 \geq \lambda_2 \geq, \ldots, \lambda_p \geq 0$. The definition of the OLS estimator of β is as follows:

$$\hat{\alpha}_{OLS} = \Pi^{-1}X'Y \tag{13}$$

Following this, MLIU estimator is given as:

$$\widehat{\alpha}_{(d)}^{ML} = G_0 G_1 \hat{\alpha}_{OLS} \tag{14}$$

where d represents MLIU Estimator's biasing parameter.

The Kibra-Lukman Estimator is given as:

$$\widehat{\alpha}_{(k)}^{KL} = D_0^{-1} D_1 \hat{\alpha}_{OLS} \tag{15}$$

Where k is the biasing parameter of Ridge Estimator

MRT Parameter is given as:

$$\widehat{\alpha}^{MRT}_{(k,d)} = G_2^{-1}\Pi\hat{\alpha}_{OLS} \tag{16}$$

Where, $G_o = \left(\Pi + I_p\right)^{-1}$, $G_1 = \left(\Pi - dI_p\right)$, $G_1 = \left(\Pi + k(1+d)I\right)$, $D_O = \left(\Pi + kI_p\right)$, $D_1 = \left(\Pi - kI_p\right)$, $D_2 = \left(\Pi + kdI_p\right)$, k> 0 and 0 <d < 1.

$$\widehat{\alpha}^{HLR}_{(k,d)} = \left(\Pi + I_p\right)^{-1}\left(\Pi - dI_p\right)\left(\Pi + k(1+d)I_p\right)^{-1}\Pi\,\hat{\alpha}_{OLS} \tag{17}$$

$$\widehat{\alpha}^{HLR}_{(k,d)} = \Pi G_0 G_1 G_2^{-1}\hat{\alpha}_{OLS} \tag{18}$$

2.2.1 Properties of the newly proposed estimator

The characteristics of the HLR Estimator are as follows: (19)

$$E\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \Pi G_0 G_1 G_2^{-1}\alpha$$

$$Bias\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \left(\Pi G_0 G_1 G_2^{-1} - I\right)\alpha \tag{20}$$

$$Var\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2 G_0 G_1 G_2^{-1}\,\Pi G_0 G_1 G_2^{-1} \tag{21}$$

$$MSEM\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2 G_0 G_1 G_2^{-1}\,\Pi G_0 G_1 G_2^{-1} + \left(\Pi G_0 G_1 G_2^{-1} - I\right)\alpha\alpha'\left(\Pi G_0 G_1 G_2^{-1} - I\right)' \tag{22}$$

2.2.2 Determination of MSE for the HLR Estimator

The MSE of OLS estimator $\widehat{\alpha}$ can be expressed as:

$$MSE\left[\widehat{\alpha}\right] = E(\widehat{\alpha} - \alpha)'(\widehat{\alpha} - \alpha) \tag{23}$$

$$= tr\left(\text{cov}(\widehat{\alpha})\right) + bias(\widehat{\alpha})'bias(\widehat{\alpha})$$

Therefore, following the submission of Lukman *et al.* (2020), the MSE of the Hybrid Liu-Ridge (HLR) Estimator can be expressed as:

$$MSE\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = trace\left[MSEM\left(\widehat{\alpha}^{HLR}_{(k,d)}\right)\right] \tag{24}$$

$$MSE\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2\sum_{i=1}^{p}\frac{\lambda_i\left(\lambda_i - d\right)^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^2} + \sum_{i=1}^{p}\frac{\left(\lambda_i\left(d - k - kd - 1\right) - k\left(1+d\right)\right)^2\alpha^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^2} \tag{25}$$

2.2.3 Selection of Shrinkage Parameter

To determine the shrinkage parameters *k* and *d* for the Hybrid Liu-Ridge (HLR) estimator, the partial derivatives of the Mean Squared Error (MSE) are taken with respect to *k* and *d*, respectively. Additionally, equation (25) can be written as equation (26):

$$MSE\left[\widehat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2\sum_{i=1}^{p}\frac{\lambda_i\left(\lambda_i - d\right)^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^2} + \sum_{i=1}^{p}\frac{\lambda_i^2\left(\lambda_i - d\right)^2\alpha^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^2} - \sum_{i=1}^{p}\frac{2\lambda_i\left(\lambda_i - d\right)\alpha^2}{\left(\lambda_i + 1\right)\left(\lambda_i + k\left(1+d\right)\right)} + \sum_{i=1}^{p}\alpha_i^2 \tag{26}$$

From equation (26),

$$\frac{\partial MSE\left[\widehat{\alpha}^{HLR}_{(k,d)}\right]}{\partial k} = \frac{-2\sigma^2\lambda_i\left(\lambda_i - d\right)^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^3} - \frac{2\lambda_i^2\left(\lambda_i - d\right)^2\left(1+d\right)\alpha^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1+d\right)\right)^3} + \frac{2\lambda_i\left(\lambda_i - d\right)\left(1+d\right)\alpha^2}{\left(\lambda_i + 1\right)\left(\lambda_i + k\left(1+d\right)\right)^2} = 0$$

Therefore, the shrinkage parameter k of the Hybrid Liu-Ridge (HLR) is expressed as:

$$k = \frac{\sigma^2\left(\lambda_i - d\right) - \lambda_i\left(1+d\right)\alpha_i^2}{\left(\lambda_i + 1\right)\left(1+d\right)\alpha_i^2} \tag{27}$$

Likewise, the shrinkage parameter d of the Hybrid Liu-Ridge (HLR) is also expressed as:

$$d = \frac{\lambda_i\left(\sigma^2 + \lambda_i\alpha_i^2\right) - \alpha_i^2\left(\lambda_i + 1\right)\left(\lambda_i + k\right)}{\sigma^2 + \left[\lambda_i + \left(\lambda_i + 1\right)k\right]\alpha_i^2} \quad (28)$$

2.2.4 Comparison of Hybrid Liu-Ridge method with Existing methods

Theoretically, the new method was compared with some of the already existing ones. Hence, to establish the superiority of the proposed method, below are the lemmas:

Lemma 1: Let $n \times n$ matrices T > 0, Y > 0 (or Y ≥ 0), then T > Y if and only if $\lambda_i\left(YT^{-1}\right) < 1$, where $\lambda_i\left(TY^{-1}\right)$ is the largest eigenvalue of matrix YT^{-1}.

Lemma 2: Let T be $n \times n$ positive definite matrix, that is, T > 0, and α be some vector, then $T - \alpha\alpha' \geq 0$ if and only if $\alpha' Y^{-1} \alpha \leq 1$.

Lemma 3: Let $\hat{\alpha}_i = D_i y, i = 1,2$ be two linear estimators of α. Suppose that $M = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, where $Cov(\hat{\alpha}_i)$,i = 1,2 denotes the covariance matrix of $\hat{\alpha}_i$ and $bi = Bias(\hat{\alpha}_i) = \left(A_i X - I\right)\alpha,\ i = 1,2$. Consequently, $\Delta\left(\hat{\alpha}_1 - \hat{\alpha}_2\right) = MSEM\left(\hat{\alpha}_1\right) - MSEM\left(\hat{\alpha}_2\right) = \sigma^2 M + b_1 b_1' - b_2 b_2' > 0$ if and only if $b_2'\left[\sigma^2 M + b_1 b_1'\right]^{-1} b_2 < 1$, where $MSEM\left(\hat{\alpha}_i\right) = Cov\left(\hat{\alpha}_i\right) + b_i b_i'$

2.2.4.1 Comparison between OLS and HLR Estimators

The OLS estimator MSEM is as follows:

$$MSEM\left[\hat{\alpha}_{OLS}\right] = \sigma^2 \Pi^{-1} \quad (29)$$

The disparity between (22) and (29)

$$MSEM\left[\hat{\alpha}_{OLS}\right] - MSEM\left[\hat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2\Pi^{-1} - \sigma^2 G_0 G_1 G_2^{-1}\Pi G_0 G_1 G_2^{-1} + (\Pi G_0 G_1 G_2^{-1} - 1)\alpha\alpha'(\Pi G_0 G_1 G_2^{-1} - 1)'$$

Let 0<d<1 and k>0. Consequently, the following theorem is true.

Theorem: This theorem states that $\hat{\alpha}^{HLR}_{(k,d)}$ is better than $\hat{\alpha}_{OLS}$ if and only if

$$\alpha'\left(\Pi G_0 G_1 G_2^{-1} - 1\right)' \sigma^2\left[(\Pi^{-1} - G_0 G_1 G_2^{-1}\Pi G_0 G_1 G_2^{-1})\right]^{-1}\left(\Pi G_0 G_1 G_2^{-1} - 1\right)\alpha < 1 \quad (30)$$

Proof: Examining the dispersion matrix variation between the $Cov\left[\hat{\alpha}_{OLS}\right]$ and $Cov\left[\hat{\alpha}^{HLR}_{(k,d)}\right]$

$$Cov\left[\alpha_{OLS}\right] - Cov\left[\hat{\alpha}^{HLR}_{(k,d)}\right] = \sigma^2\left[(\Pi^{-1} - G_0 G_1 G_2^{-1}\Pi G_0 G_1 G_2^{-1})\right] \quad (31)$$

$$= \sigma^2\, diag\left\{\frac{1}{\lambda_i} - \frac{\lambda_i\left(\lambda_i - d\right)^2}{\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1 + d\right)\right)^2}\right\}_{i=1}^{p} \quad (32)$$

$\Pi^{-1} - G_0 G_1 G_2^{-1}\Pi G_0 G_1 G_2^{-1}$ Will become pdf if and only if $\left(\lambda_i + 1\right)^2\left(\lambda_i + k\left(1 + d\right)\right)^2 - \lambda_i^3\left(\lambda_i - d\right)^2 > 0$. By third lemma, the proof has been accomplished

2.2.4.2 A comparison between the MLIU and HLR Estimators.

The covariance matrix, bias vector, and MSEM of the Modified Liu Estimator are as follows:

$$Bias\left[\hat{\alpha}^{MLIU}_{(d)}\right] = (G_0 G_1 - 1)\alpha \quad (33)$$

$$Var\left[\hat{\alpha}^{MLIU}_{(d)}\right] = \sigma^2 G_0 G_1 \Pi^{-1} G_0 G_1 \quad (34)$$

$$MSEM\left[\hat{\alpha}^{MLIU}_{(d)}\right] = \sigma^2 G_0 G_1 \Pi^{-1} G_0 G_1 + (G_0 G_1 - 1)\alpha\alpha'(G_0 G_1 - 1)' \quad (35)$$

Theorem: This theorem states that $\hat{\alpha}_{(k,d)}^{HLR}$ is better than $\hat{\alpha}_{(d)}^{ML}$ if and only if $MSEM\left[\hat{\alpha}_{OLS}\right] - MSEM\left[\hat{\alpha}_{(k,d)}^{HLR}\right] > 0$ if and only if

$$\alpha' \left(\Pi G_0 G_1 G_2^{-1} - 1\right)' \left[\sigma^2 \left(G_0 G_1 \Pi^{-1} G_0 G_1 - G_0 G_1 G_2^{-1} \Pi G_0 G_1 G_2^{-1}\right) + (G_0 G_1 - 1)\alpha\alpha'(G_0 G_1 - 1)'\right]^{-1} \left(\Pi G_0 G_1 G_2^{-1} - 1\right)\alpha < 1$$

Proof: Examining the dispersion matrix variation between the $Cov\left[\hat{\alpha}_{(d)}^{MLIU}\right]$ and $Cov\left[\hat{\alpha}_{(k,d)}^{HLR}\right]$

$$D_f = \sigma^2 \left(G_0 G_1 \Pi^{-1} G_0 G_1 - G_0 G_1 G_2^{-1} \Pi G_0 G_1 G_2^{-1}\right)$$

$$D_f = \sigma^2 (\Pi + I_p)^{-1} (\Pi - dI_p) \Pi^{-1} (\Pi + I_p)^{-1} (\Pi - dI_p)' - \sigma^2 (\Pi + I_p)^{-1} (\Pi - dI_p)(\Pi + k(1+d)I_p)^{-1} \Pi (\Pi + I_p)^{-1} (\Pi - dI_p)(\Pi + k(1+d)I_p)^{-1}$$

$$D_f = \sigma^2 diag\left\{\frac{(\lambda_i - d)^2}{\lambda_i(\lambda_i+1)^2} - \frac{\lambda_i(\lambda_i - d)^2}{(\lambda_i+1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (36)$$

Will become pdf if and only if $(\lambda_i - d)^2 (\lambda_i + k(1+d))^2 - \lambda_i^2 (\lambda_i - d)^2 > 0$ for 0<d<1 and k>0, it was observed that$(\lambda_i - d)^2 (\lambda_i + k(1+d))^2 - \lambda_i^2 (\lambda_i - d)^2 > 0$. By third lemma, the proof has been accomplished.

2.2.4.3 Comparison between KL and HLR

The covariance matrix, bias vector, and MSEM of K- L Estimator are as follows:

$$Bias\left[\hat{\alpha}_{(k)}^{KL}\right] = (D_0^{-1} D_1 - 1)\alpha$$

$$Var\left[\hat{\alpha}_{(k)}^{KL}\right] = \sigma^2 D_0^{-1} D_1 \Pi^{-1} D_0^{-1} D_1 \quad (38)$$

$$MSEM\left[\hat{\alpha}_{(k)}^{KL}\right] = \sigma^2 D_0^{-1} D_1 \Pi^{-1} D_0^{-1} D_1 + (D_0^{-1} D_1 - 1)\alpha\alpha'(D_0^{-1} D_1 - 1)' \quad (39)$$

Theorem: This theorem states that the proposed estimator $\hat{\alpha}_{(k,d)}^{HLR}$ is superior to $\hat{\alpha}_{(k)}^{KL}$ if and only if $MSEM\left[\hat{\alpha}_{(k)}^{KL}\right] - MSEM\left[\hat{\alpha}_{(k,d)}^{HLR}\right] > 0$ if and only if

$$\alpha' \left(\Pi G_0 G_1 G_2^{-1} - 1\right)' \begin{bmatrix} \sigma^2 \left(D_0^{-1} D_1 \Pi^{-1} D_0^{-1} D_1 - G_0 G_1 G_2^{-1} \Pi G_0 G_1 G_2^{-1}\right) + \\ (D_0^{-1} D_1 - 1)\alpha\alpha'(D_0^{-1} D_1 - 1)' \end{bmatrix}^{-1} \left(\Pi G_0 G_1 G_2^{-1} - 1\right)\alpha < 1 \quad (40)$$

Proof: Examining the dispersion matrix variation between the $Cov\left[\hat{\alpha}_{(k)}^{KL}\right]$ and $Cov\left[\hat{\alpha}_{(k,d)}^{HLR}\right]$

$$D_f = \sigma^2 \left(D_0^{-1} D_1 \Pi^{-1} D_0^{-1} D_1 - G_0 G_1 G_2^{-1} \Pi G_0 G_1 G_2^{-1}\right)$$

$$D_f = \sigma^2 \left(\Pi + kI_p\right)^{-1} \left(\Pi - kI_p\right) \Pi^{-1} \left(\Pi + kI_p\right)^{-1} \left(\Pi - kI_p\right)' - \sigma^2 \left(\Pi + I_p\right)^{-1} \left(\Pi - dI_p\right) \left(\Pi + k(1+d)I_p\right)^{-1} \Pi \left(\Pi + I_p\right)^{-1} \left(\Pi - dI_p\right) \left(\Pi + k(1+d)I_p\right)^{-1}$$

$$D_f = \sigma^2 diag\left\{\frac{(\lambda_i - k)^2}{\lambda_i^2(\lambda_i+k)^2} - \frac{\lambda_i(\lambda_i - d)^2}{(\lambda_i+1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (41)$$

Will become pdf if and only if $(\lambda_i + 1)^2 (\lambda_i - k)^2 (\lambda_i + k(1+d))^2 - \lambda_i^3 (\lambda_i + k)^2 (\lambda_i - d)^2 > 0$. For 0<d<1 and k>o, it was observed that$(\lambda_i + 1)^2 (\lambda_i - k)^2 (\lambda_i + k(1+d))^2 - \lambda_i^3 (\lambda_i + k)^2 (\lambda_i - d)^2 > 0$. By third lemma, the proof has been accomplished.

2.2.4.4 Comparison between MRT and HLR Estimators

The covariance matrix, bias vector, and MSEM of the Modified ridge type estimator $\hat{\alpha}_{(k,d)}^{MRT}$ are given below:

$$Bias\left[\hat{\alpha}_{(k,d)}^{MRT}\right]=\left(\prod G_2^{-1}-1\right)\alpha \quad (42)$$

$$Var\left[\hat{\alpha}_{(k,d)}^{MRT}\right]=\sigma^2 G_2^{-1}\prod G_0^{-1} \quad (43)$$

$$MSEM\left[\hat{\alpha}_{(k,d)}^{MRT}\right]=\sigma^2 G_2^{-1}\prod G_0^{-1}+\left(\prod G_2^{-1}-1\right)\alpha\alpha'\left(\prod G_2^{-1}-1\right)' \quad (44)$$

Let k>0 and o<d<1. Therefore, the following theorem holds.

Theorem: This theorem states that the proposed estimator $\hat{\alpha}_{(k,d)}^{HLR}$ is superior to $\hat{\alpha}_{(k,d)}^{MRT}$ if and only if $MSEM\left[\hat{\alpha}_{(k,d)}^{MRT}\right]-MSEM\left[\hat{\alpha}_{(k,d)}^{HLR}\right]>0$

$$\alpha'\left(\Pi G_0 G_1 G_2^{-1}-1\right)'\begin{bmatrix}\sigma^2\left(G_2^{-1}\Pi^{-1}G_2^{-1}-G_0G_1G_2^{-1}\Pi G_0G_1G_2^{-1}\right)+\\ \left(\Pi G_2^{-1}-1\right)\alpha\alpha'\left(\Pi G_2^{-1}-1\right)'\end{bmatrix}^{-1}\left(\Pi G_0G_1G_2^{-1}-1\right)\alpha<1$$

Proof: Examining the dispersion matrix variation between the $MSEM\left[\hat{\alpha}_{(k,d)}^{MRT}\right]-MSEM\left[\hat{\alpha}_{(k,d)}^{HLR}\right]>0$

$$D_f==\sigma^2\left(G_2^{-1}\Pi^{-1}G_2^{-1}-G_0G_1G_2^{-1}\Pi G_0G_1G_2^{-1}\right)$$

$$D_f=\sigma^2\left(\Pi+k(1+d)I_p\right)^{-1}\Pi\left(\Pi+k(1+d)I_p\right)^{-1}-\sigma^2\left(\Pi+I_p\right)^{-1}\left(\Pi-dI_p\right)\left(\Pi+k(1+d)I_p\right)^{-1}\Pi\left(\Pi+I_p\right)^{-1}\left(\Pi-dI_p\right)\left(\Pi+k(1+d)I_p\right)^{-1}$$

$$D_f=\sigma^2 diag\left\{\frac{\lambda_i}{\left(\lambda_i+k\left(1+d\right)\right)^2}-\frac{\lambda_i\left(\lambda_i-d\right)^2}{\left(\lambda_i+1\right)^2\left(\lambda_i+k\left(1+d\right)\right)^2}\right\}_{i=1}^{p} \quad (45)$$

Will become pdf if and only if $\lambda_i\left(\lambda_i+1\right)^2-\lambda_i\left(\lambda_i-d\right)^2>0$, for 0<d<1 and k>0, it was observed that $\lambda_i\left(\lambda_i+1\right)^2-\lambda_i\left(\lambda_i-d\right)^2>0$. By third lemma, the proof has been accomplished.

2.2.5 Monte Carlo Experiment

R statistical programming was used to run the Monte Carlo simulation for this investigation. OLS, KL, DK, Ridge, Liu, NTP, MRT, and Hybrid Liu-Ridge (HLR) were among the estimators whose performance was assessed. Lukman *et al*. (2020) and other researchers in the field used the equation shown in (46), which was used to produce the exogenous variables.

$$x_{ij}=(1-\vartheta^2)^{\frac{1}{2}}z_{ij}+\vartheta z_{ip+1}, i = 1,2,\ldots,n, j\ 1,2,\ldots,p. \quad (46)$$

where ϑ denotes the correlation between any two exogenous variables and z_{ij} is an independent standard normal pseudo-random number. To show the degrees of correlations between the explanatory variables, four (4) levels of various correlations (ϑ) 0.85, 0.9, 0.95, and 0.99 were taken into consideration. Concurrently, there are p = 3 exogenous variables, which are represented in a standardized manner. Similarly, the following equations were used to create the response variable:

$$y=\beta_0+\beta_1x_1+\ldots+\beta_px_{ip}+e_i, i=1, \ldots, p \quad (47)$$

where $e_i \sim iidN(0,\sigma^2)$ the values of β were selected to meet the limitations $\beta'\beta=1$ proposed by Lukman *et al*. (2020), with zero intercept assumed for the model in (47). With error variances of (1, 5, and 10), the simulation research was repeated 1000 times for sample sizes (n) = 10, 20, 30, 40, 50, 100, 250, and 500, respectively. Additionally, each replicate's predicted MSE for each estimator was determined using equation (48).

$$MSE(\beta^*)=\frac{1}{1000}\sum_{i=1}^{1000}(\beta^*-\beta)^2 \quad (48)$$

In the same vein, the shrinkage parameters k and d used in this study are the parameters suggested and used by Kibria and Lukman (2020) among other researchers which were estimated to be 0.0012 and 0.5 respectively.

2.2.6 Criterion for Investigation of Proposed Biasing Parameter Estimators

The criterion used as the yardstick for the selection of the best parameter estimators:

2.2.6.1 Use of Mean Square Error (MSE): In this study, the estimator that its rank of MSE is approximately less than or equal to 5 was considered the best estimator using MSE as criterion as suggested by Adejumo *et al.* (2024).

3.0 Result and Discussion

3.1 Simulation Results

Table 1: Simulation results of Estimated MSE

			OLS	RIDGE	LIU	KL	MRT	NTP	DK	HLR
		ϑ=0.85	2.76317	2.75324	1.44545	2.74333	2.7582	2.74829	2.73347	**0.22957**
	sig=1	ϑ=0.9	4.00713	3.98557	1.84445	3.96408	3.99634	3.97487	3.94276	**0.24123**
		ϑ=0.95	7.74754	7.66402	2.88212	7.58101	7.70572	7.62283	7.49926	**0.54164**
		ϑ=0.99	37.6987	35.7267	10.484	33.8135	36.7054	34.8043	32.0383	**7.00143**
		ϑ=0.85	69.0791	68.8309	36.1138	68.5831	68.9549	68.7073	68.3367	**5.57091**
n=10	sig=5	ϑ=0.9	100.178	99.6383	45.9484	99.1001	99.908	99.3703	98.5661	**5.29738**
		ϑ=0.95	193.689	191.601	72.0326	189.525	192.643	190.571	187.482	**13.3888**
		ϑ=0.99	942.468	893.167	262.08	845.339	917.634	870.106	800.956	**174.892**
		ϑ=0.85	276.317	275.324	144.45	274.333	275.82	274.829	273.347	**22.2565**
	sig=10	ϑ=0.9	400.713	398.553	183.742	396.4	399.632	397.48	394.263	**21.0908**
		ϑ=0.95	774.754	766.402	288.126	758.101	770.572	762.283	749.926	**53.5308**
		ϑ=0.99	3769.87	3572.67	1048.32	3381.35	3670.54	3480.43	3203.83	**699.548**
		ϑ=0.85	0.97957	0.97826	0.6835	0.97696	0.97892	0.97761	0.97565	**0.28637**
	sig=1	ϑ=0.9	1.42064	1.41773	0.88229	1.41482	1.41919	1.41628	1.41193	**0.25103**
		ϑ=0.95	2.7471	2.73558	1.36727	2.72409	2.74134	2.72985	2.71267	**0.18583**
		ϑ=0.99	13.3694	13.087	4.26398	12.8081	13.2278	12.9496	12.5375	**1.55847**
		ϑ=0.85	24.4893	24.4566	17.0683	24.4239	24.4729	24.4402	24.3912	**7.0261**
n=20	sig=5	ϑ=0.9	35.5161	35.4433	22.0512	35.3706	35.4797	35.407	35.2981	**6.22405**
		ϑ=0.95	68.6775	68.3895	34.1629	68.1022	68.5334	68.2463	67.8166	**4.52661**
		ϑ=0.99	334.234	327.175	106.58	320.202	330.694	323.74	313.436	**38.862**
		ϑ=0.85	97.9572	97.8262	68.2686	97.6954	97.8917	97.7609	97.5649	**28.0814**
	sig=10	ϑ=0.9	142.064	141.773	88.2041	141.482	141.919	141.628	141.193	**24.8894**
		ϑ=0.95	274.71	273.558	136.646	272.409	274.134	272.985	271.267	**18.0853**
		ϑ=0.99	1336.94	1308.7	426.314	1280.81	1322.78	1294.96	1253.75	**155.434**
		ϑ=0.85	0.43739	0.43721	0.37727	0.43703	0.4373	0.43712	0.43685	**0.27285**
	sig=1	ϑ=0.9	0.62657	0.62618	0.50535	0.62578	0.62638	0.62598	0.62539	**0.30623**
		ϑ=0.95	1.19591	1.19434	0.81784	1.19277	1.19513	1.19356	1.1912	**0.28961**
		ϑ=0.99	5.75644	5.71745	2.31929	5.67859	5.73693	5.6981	5.64006	**0.12516**
		ϑ=0.85	10.9347	10.9303	9.4295	10.9258	10.9325	10.928	10.9213	**6.79199**
n=30	sig=5	ϑ=0.9	15.6643	15.6544	12.6299	15.6445	15.6594	15.6495	15.6346	**7.62985**
		ϑ=0.95	29.8978	29.8585	20.4445	29.8193	29.8782	29.8389	29.7801	**7.21523**
		ϑ=0.99	143.911	142.936	57.9813	141.965	143.423	142.453	141.002	**3.10304**
		ϑ=0.85	43.7389	43.721	37.7182	43.7032	43.73	43.7121	43.6853	**27.1656**
	sig=10	ϑ=0.9	62.6574	62.6177	50.5184	62.5781	62.6376	62.5979	62.5385	**30.5145**
		ϑ=0.95	119.591	119.434	81.7783	119.277	119.513	119.356	119.12	**28.8592**
		ϑ=0.99	575.644	571.745	231.926	567.859	573.693	569.81	564.007	**12.4093**
		ϑ=0.85	0.34442	0.3443	0.30251	0.34418	0.34436	0.34424	0.34406	**0.22845**
	sig=1	ϑ=0.9	0.49675	0.49648	0.41099	0.49621	0.49662	0.49635	0.49594	**0.26697**
		ϑ=0.95	0.95472	0.95364	0.68125	0.95256	0.95418	0.9531	0.95149	**0.2836**
		ϑ=0.99	4.62163	4.59485	1.97495	4.56815	4.60823	4.58155	4.54165	**0.07745**
		ϑ=0.85	8.61059	8.60754	7.56041	8.60448	8.60907	8.60601	8.60143	**5.69474**
n=40	sig=5	ϑ=0.9	12.4188	12.412	10.274	12.4052	12.4154	12.4086	12.3984	**6.66558**
		ϑ=0.95	23.868	23.841	17.0293	23.814	23.8545	23.8275	23.7871	**7.07518**
		ϑ=0.99	115.541	114.871	49.372	114.204	115.206	114.539	113.541	**1.92115**
		ϑ=0.85	34.4424	34.4302	30.2411	34.4179	34.4363	34.424	34.4057	**22.776**

	sig=10	$\vartheta=0.9$	49.6753	49.6481	41.0961	49.6209	49.6617	49.6345	49.5938	**26.6615**
		$\vartheta=0.95$	95.4718	95.364	68.1168	95.2561	95.4179	95.3101	95.1485	**28.2981**
		$\vartheta=0.99$	462.163	459.485	197.488	456.815	460.823	458.155	454.165	**7.68156**
n=50	sig=1	$\vartheta=0.85$	0.38169	0.38153	0.32831	0.38137	0.38161	0.38145	0.38121	**0.23538**
		$\vartheta=0.9$	0.55689	0.55654	0.44832	0.55618	0.55672	0.55636	0.55582	**0.27042**
		$\vartheta=0.95$	1.08323	1.08181	0.74389	1.08038	1.08252	1.08109	1.07896	**0.27042**
		$\vartheta=0.99$	5.2963	5.2609	2.17275	5.22563	5.27858	5.24334	5.19067	**0.11408**
	sig=5	$\vartheta=0.85$	9.54229	9.53829	8.2064	9.53428	9.54029	9.53629	9.53028	**5.87772**
		$\vartheta=0.9$	13.9224	13.9134	11.2085	13.9045	13.9179	13.9089	13.8955	**6.75991**
		$\vartheta=0.95$	27.0808	27.0451	18.5959	27.0095	27.0629	27.0273	26.974	**6.75428**
		$\vartheta=0.99$	132.408	131.522	54.3175	130.641	131.965	131.084	129.767	**2.84623**
	sig=10	$\vartheta=0.85$	38.1692	38.1532	32.825	38.1371	38.1612	38.1451	38.1211	**23.5089**
		$\vartheta=0.9$	55.6894	55.6536	44.8345	55.6178	55.6715	55.6357	55.582	**27.0407**
		$\vartheta=0.95$	108.323	108.181	74.3832	108.038	108.252	108.109	107.896	**27.0153**
		$\vartheta=0.99$	237.834	236.567	103.561	235.303	237.2	235.937	234.048	**3.95439**
n=100	sig=1	$\vartheta=0.85$	0.16861	0.16858	0.15623	0.16855	0.1686	0.16856	0.16851	**0.13306**
		$\vartheta=0.9$	0.24621	0.24613	0.21985	0.24606	0.24617	0.24609	0.24598	**0.17205**
		$\vartheta=0.95$	0.47926	0.47896	0.38771	0.47866	0.47911	0.47881	0.47837	**0.23569**
		$\vartheta=0.99$	2.34452	2.33714	1.22142	2.32979	2.34083	2.33347	2.32246	**0.08989**
	sig=5	$\vartheta=0.85$	4.21529	4.21446	3.90559	4.21363	4.21487	4.21405	4.2128	**3.32585**
		$\vartheta=0.9$	6.15514	6.15329	5.49608	6.15144	6.15422	6.15236	6.14958	**4.30068**
		$\vartheta=0.95$	11.9814	11.974	9.69276	11.9666	11.9777	11.9703	11.9592	**5.89158**
		$\vartheta=0.99$	58.6129	58.4286	30.5353	58.2447	58.5207	58.3368	58.0615	**2.24657**
	sig=10	$\vartheta=0.85$	16.8612	16.8578	15.6224	16.8545	16.8595	16.8562	16.8512	**13.3033**
		$\vartheta=0.9$	24.6206	24.6132	21.9844	24.6057	24.6169	24.6095	24.5983	**17.2027**
		$\vartheta=0.95$	47.9255	47.8959	38.771	47.8663	47.9107	47.8811	47.8368	**23.5662**
		$\vartheta=0.99$	234.452	233.714	122.141	232.979	234.083	233.347	232.246	**8.98616**
n=250	sig=1	$\vartheta=0.85$	0.05686	0.05686	0.05555	0.05685	0.05686	0.05685	0.05685	**0.053**
		$\vartheta=0.9$	0.0824	0.08239	0.07952	0.08238	0.08239	0.08239	0.08237	**0.07394**
		$\vartheta=0.95$	0.15911	0.15908	0.14822	0.15905	0.1591	0.15907	0.15903	**0.12766**
		$\vartheta=0.99$	0.77318	0.77247	0.5739	0.77175	0.77282	0.77211	0.77104	**0.26562**
	sig=5	$\vartheta=0.85$	1.42146	1.42138	1.38873	1.42129	1.42142	1.42134	1.42121	**1.32453**
		$\vartheta=0.9$	2.0599	2.05972	1.98809	2.05954	2.05981	2.05963	2.05936	**1.84857**
		$\vartheta=0.95$	3.97781	3.97709	3.70542	3.97637	3.97745	3.97673	3.97565	**3.19091**
		$\vartheta=0.99$	19.3295	19.3116	14.347	19.2938	19.3206	19.3027	19.2759	**6.63968**
	sig=10	$\vartheta=0.85$	5.68583	5.6855	5.55488	5.68518	5.68566	5.68534	5.68485	**5.29798**
		$\vartheta=0.9$	8.23959	8.23887	7.95238	8.23814	8.23923	8.23851	8.23742	**7.39431**
		$\vartheta=0.95$	15.9112	15.9084	14.8216	15.9055	15.9098	15.9069	15.9026	**12.7634**
		$\vartheta=0.99$	77.3182	77.2466	57.3878	77.1751	77.2824	77.2109	77.1036	**26.5583**
n=500	sig=1	$\vartheta=0.85$	0.02846	0.02846	0.02813	0.02846	0.02846	0.02846	0.02845	**0.02747**
		$\vartheta=0.9$	0.04136	0.04136	0.04063	0.04136	0.04136	0.04136	0.04136	**0.03919**
		$\vartheta=0.95$	0.08012	0.08012	0.07728	0.08011	0.08012	0.08011	0.0801	**0.07175**
		$\vartheta=0.99$	0.3904	0.39022	0.3305	0.39004	0.39031	0.39013	0.38986	**0.22577**
	sig=5	$\vartheta=0.85$	0.71142	0.7114	0.70318	0.71138	0.71141	0.71139	0.71136	**0.68685**
		$\vartheta=0.9$	1.03401	1.03397	1.01575	1.03392	1.03399	1.03394	1.03388	**0.97973**
		$\vartheta=0.95$	2.00308	2.0029	1.932	2.00272	2.00299	2.00281	2.00254	**1.79382**
		$\vartheta=0.99$	9.7599	9.75544	8.26244	9.75098	9.75767	9.75321	9.74652	**5.64432**
	sig=10	$\vartheta=0.85$	2.8457	2.84562	2.81273	2.84554	2.84566	2.84558	2.84546	**2.74741**
		$\vartheta=0.9$	4.13604	4.13586	4.06298	4.13568	4.13595	4.13577	4.13551	**3.91893**
		$\vartheta=0.95$	8.01232	8.01161	7.72801	8.01089	8.01196	8.01125	8.01018	**7.17529**
		$\vartheta=0.99$	39.0396	39.0218	33.0498	39.0039	39.0307	39.0128	38.9861	**22.5773**

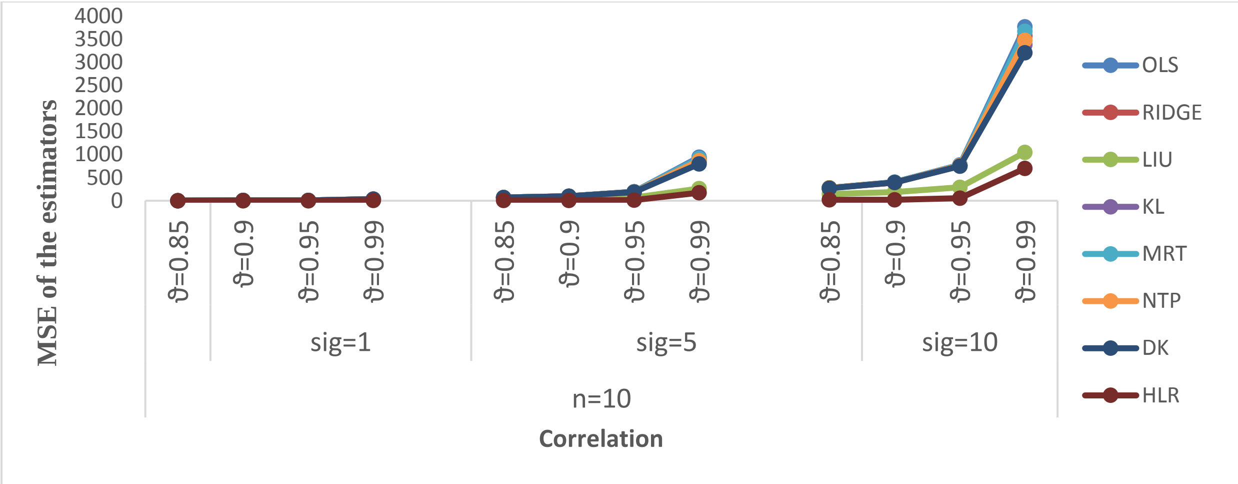

Figure 1: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =10

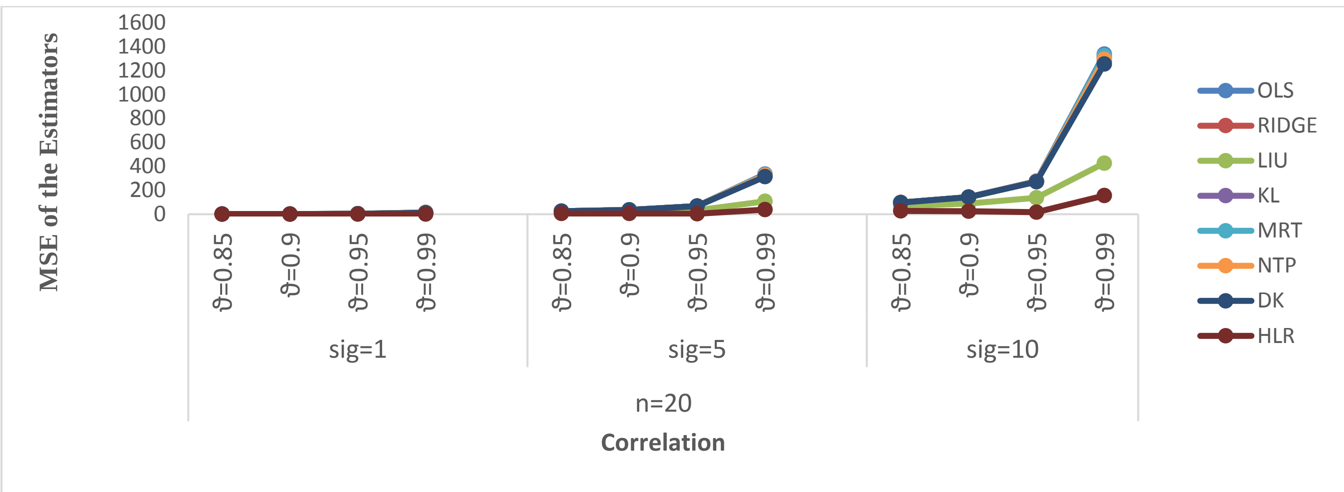

Figure 2 A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =20

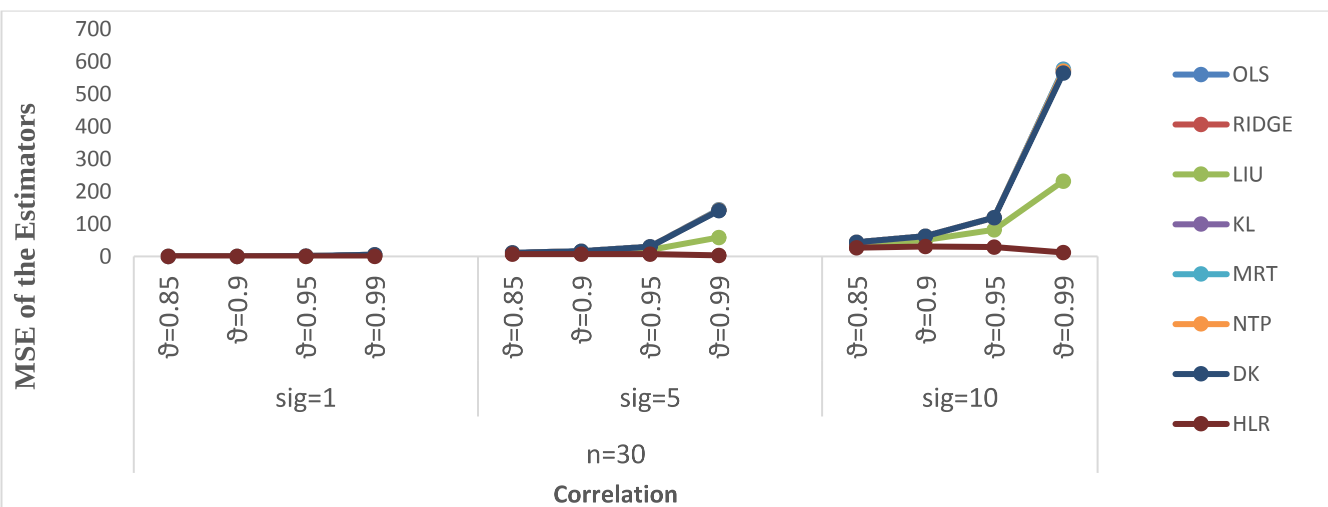

Figure 3: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =30

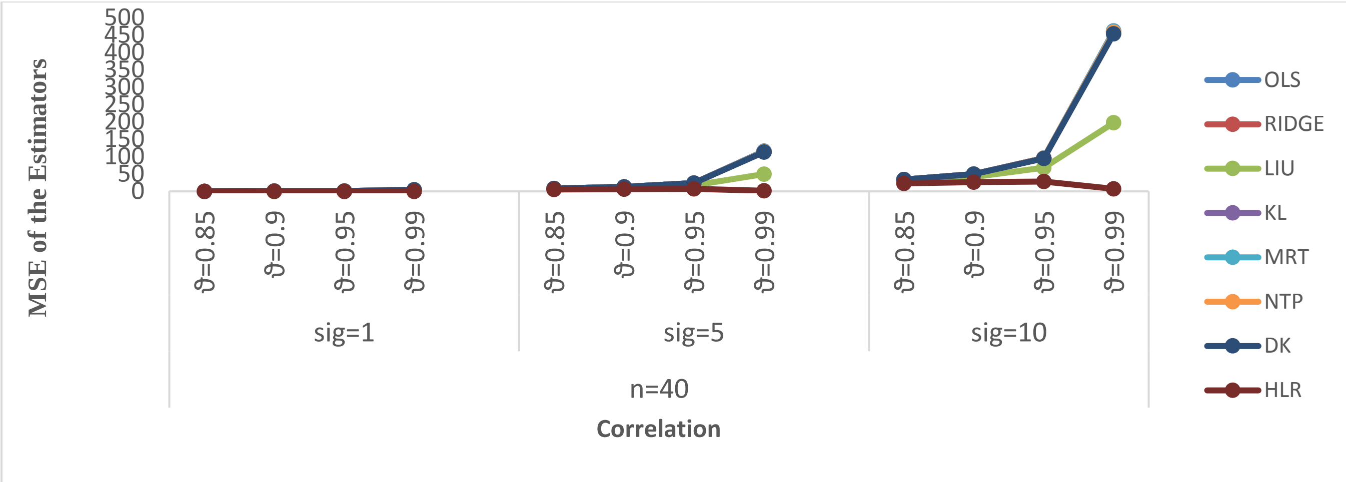

Figure 4: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =40

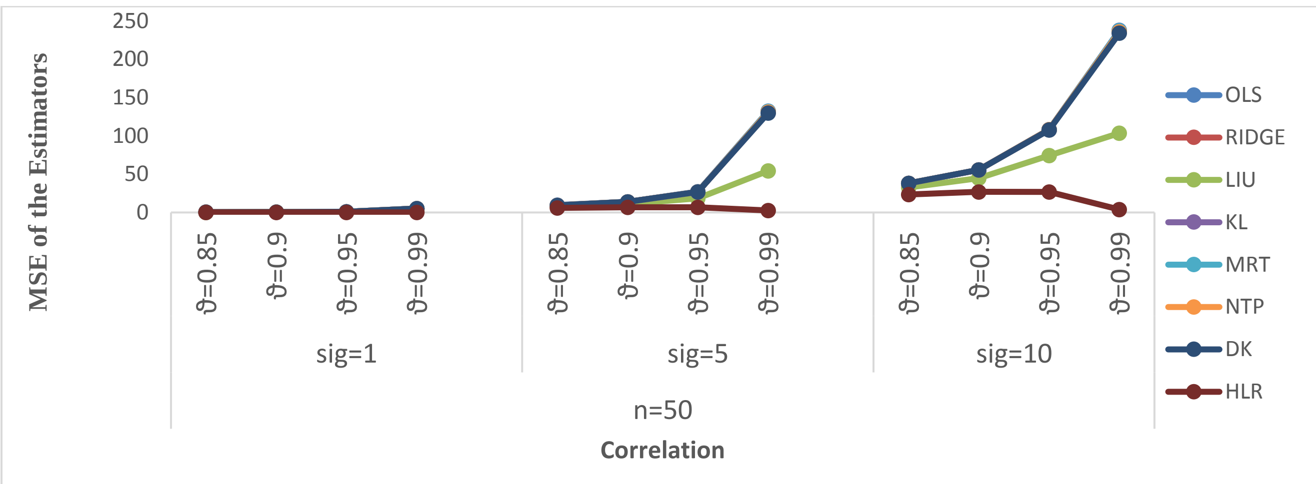

Figure 5: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =50

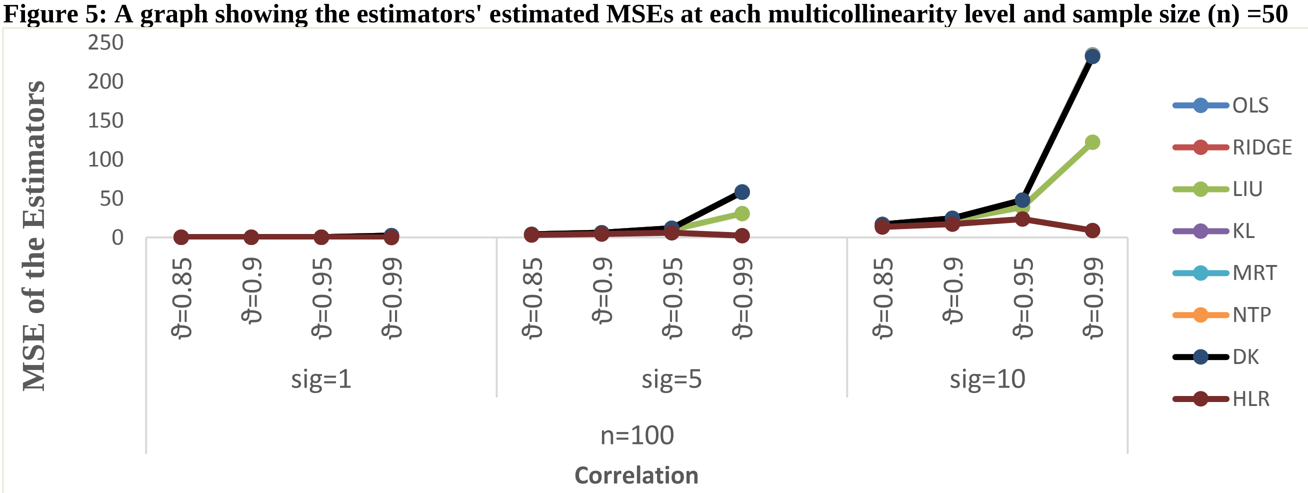

Figure 6: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =100

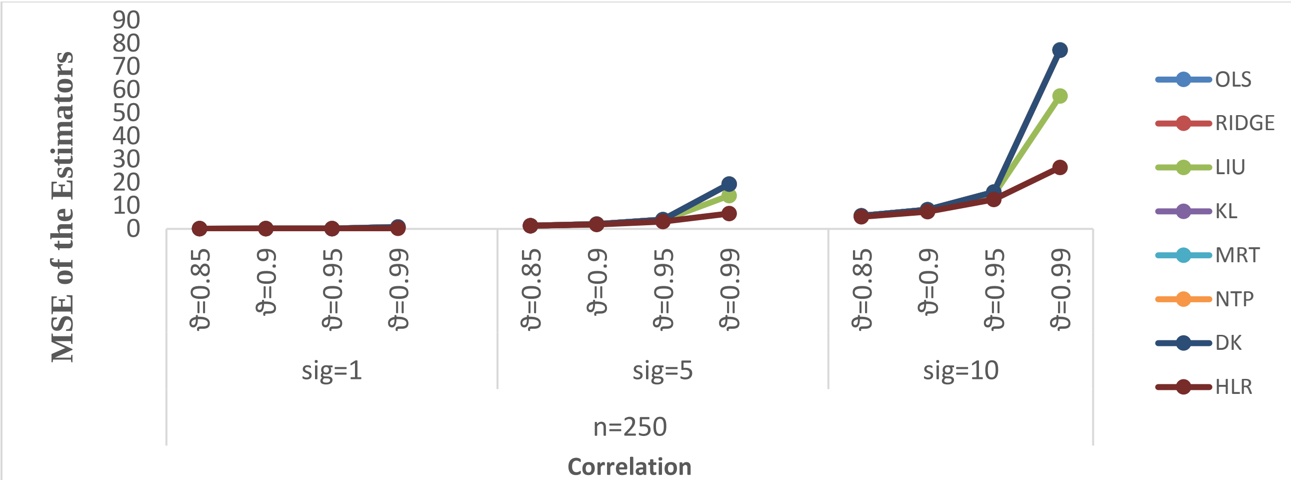

Figure 7: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =250

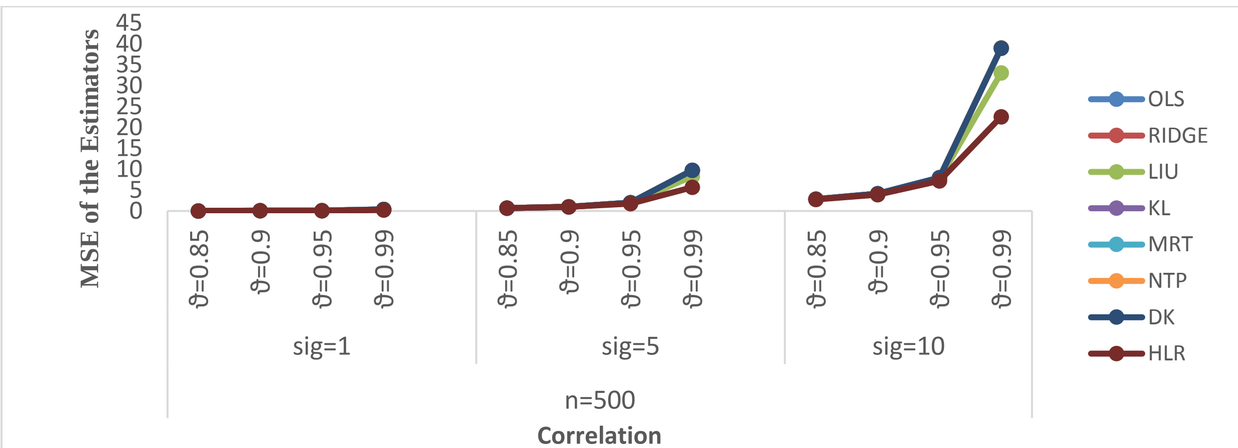

Figure 8: A graph showing the estimators' estimated MSEs at each multicollinearity level and sample size (n) =500

3.2 Discussion

Regarding the simulation outcomes that are shown in Table 1 and illustrated graphically in Figures 1–8, the following is a list of comments:

(i) As anticipated, OLS performed awfully as multicollinearity increased.

(ii) As increases in error variances (sig) and multicollinearity levels (ϑ) occur, the MSEs of the estimators under consideration also rise.

(iii) Error variance has great effect on the mean Square Errors (MSEs). It was observed that as the error variances (1, 5 and 10) increases, the MSEs of the estimator increases.

(iv) The MSEs of the estimators drop with increasing sample size (n).

Therefore, Hybrid Liu-Ridge estimator has the minimum mean square errors (MSE) at all levels of specifications considered in the study.

3.3 Application to real-life data

This study uses real-world data from economics and agriculture, as adapted by Hussein and Abdalla (2012).

3.3.1. Application to Agricultural Data

The linear model for agricultural data used is shown below.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \tag{49}$$

where y represents the imported capital commodities, x_1 the value of the imported intermediate, the value of the imported raw materials (x_2), and the product value in the manufacturing sector (x_3). According to Hussein and Abdulla (2012), the variance inflation factor (VIF) values, which were calculated to be (128.29, 103.43, and 70.87), indicate that the data suffers from multicollinearity.

Table 2: MSEs and regression coefficients of estimators with agricultural data

Coefficients	α_{OLS}	α_{RE}	α_{LIU}	α_{KL}	α_{NTP}	α_{MRT}	α_{DK}	α_{HLR}
β_0	208.8853	208.8631	218.1294	208.8409	218.0788	208.852	208.8187	199.6137
β_1	0.612954	0.61314	0.536651	0.613326	0.535841	0.613233	0.613512	0.690722
β_2	1.25626	1.256189	1.286481	1.256118	1.285627	1.256153	1.256047	1.226643
β_3	-1.22126	-1.22131	-1.20833	-1.22137	-1.19999	-1.22134	-1.22142	-1.24272
MSE	1850.482	1850.089	8.54E+19	83504.68	1702.909	1849.893	1849.304	**1426.719**

3.3.2. Application to Economics data

For economics data, the regression equation is defined as:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_6 x_6 \quad (50)$$

Six independent variables are included in the data set: x_1 (price deflator), x_2 (gross national product), x_3 (unemployment), x_4 (armed forces), x_5 (population), and x_6 (year). The dependent variable, y, is employment. The data had multicollinearity issues, according to the variance inflation factor (VIF), which showed that the independent variables have Variance Inflation Factor (VIF) values of 132.59, 968.05, 11.70, 3.48, and 389.62 respectively.

Table 3: MSEs and regression coefficients of estimators with Economics data

Coefficients	α_{OLS}	α_{RE}	α_{LIU}	α_{KL}	α_{NTP}	α_{MRT}	α_{DK}	α_{HLR}
β_0	-3482.26	0.017077	-1.5E+10	3482.293	-141.529	0.130591	3482.52	967330.5
β_1	0.015062	-0.05293	290294.7	-0.12091	-0.05016	-0.05293	-0.12092	-18.9388
β_2	-0.03582	0.071054	-455956	0.177927	0.06671	0.071057	0.177934	29.759
β_3	-0.0202	-0.00424	-68111	0.011728	-0.00489	-0.00424	0.011729	4.430727
β_4	-0.01033	-0.00573	-19649.7	-0.00112	-0.00591	-0.00573	-0.00112	1.273906
β_5	-0.0511	-0.414	1548823	-0.77689	-0.39925	-0.41401	-0.77692	-101.221
β_6	1.829151	0.048397	7595832	-1.73236	0.120781	0.048339	-1.73247	-494.622
MSE	792848.8	3.40256	4.13E+14	965.4497	3.383129	3.398549	792644.5	**3.361474**

In this study the value of parameter *k* and *d* used is 0.0012 and 0.5 respectively which was adopted by several authors like Kibra and Lukman (2020) among others. The real life data results presented in Table 3 and 4 show the estimated MSE values and parameters of the newly proposed estimator and already existing ones. Hence, from the Tables, the MSE of the Hybrid Liu-|Ridge estimator is the lowest when compared to the MSEs of the other estimators, as indicated by the highlighted values in Table 3 and 4.

4.0 Conclusion

In order to address the problem of multicollinearity in linear regression models, this work presents a novel biased estimating method. The proposed estimator was compared with some already existing estimators through theoretical analysis. Additionally, a simulation-based study was conducted to evaluate the proposed estimator's performance in comparison to the current estimators. The findings from the simulation study, theoretical comparison, and real-world data application show that Hybrid Liu-Ridge estimator produces better results in terms of minimum (MSE). It is expected that this study will give academics from a variety of field’s insightful information.

5.0 Recommendations

Researchers and practitioners in statistical modeling should adopt the Hybrid Liu-Ridge (HLR) estimator as a reliable alternative for addressing multicollinearity in linear regression models. The Hybrid Liu-Ridge (HLR) estimator is recommended for use in upcoming studies in light of the results.

5.1 Practical benefits of the newly proposed estimator

i. **Enhanced Stability and Reliability**: The HLR estimator substantially reduces the instability induced by multicollinearity in regression models, resulting in more dependable parameter estimations than classic approaches such as OLS, Ridge Regression, and Liu Estimator.
ii. **Optimized Performance**: Because the HLR estimator consistently minimizes Mean Squared Error (MSE) across different levels of multicollinearity, error variances, and sample sizes, it provides greater prediction accuracy and is robust in a variety of scenarios.
iii. **Versatility across Applications**: The HLR estimator's practical relevance is demonstrated by its validation using real-world data in economics and agriculture, demonstrating its usefulness in a variety of fields that depend on regression analysis.
iv. **Adaptability to Diverse Conditions**: Monte Carlo simulations were used to thoroughly test the estimator's performance, demonstrating its dependability and flexibility even in difficult situations with small sample sizes or high multicollinearity.
v. **Advancement in Methodology**: By combining the best features of both the Modified Ridge Type (MRT) and Modified Liu (MLIU) estimators, the HLR estimator offers a fresh approach to a persistent problem in statistical modeling.
vi. **Usability for Practitioners**: The HLR estimator provides researchers and practitioners with a more efficient and useful tool for evaluating complicated datasets with multicollinearity by overcoming significant shortcomings of current biased estimators.

These advantages highlight the estimator's capacity to improve statistical modeling and produce trustworthy outcomes in both academic and practical settings.

5.2 Potential Limitations of the Proposed HLR Estimator

i. **Dependence on Biasing Parameters**: The performance of the proposed HLR estimator is greatly impacted by the careful choice of shrinkage parameters *k* and *d*. Although the study employs parameters that have been proposed by previous research (k=0.0012, d=0.5), the selection procedure might not be always the best and could change depending on the features of the dataset. The development of adaptive techniques for parameter selection may be the main focus of future research.
ii. **Limited Scope of Validation**: The HLR estimator's performance in broader applications, such biomedical or engineering datasets, has not yet been investigated, despite its validation using simulated and real-world datasets (economic and agricultural). Its resilience across many sectors would be ensured by broadening the validation's scope.
iii. **Sensitivity to Data Characteristics**: Extreme data outliers may cause the proposed estimator to become sensitive. Although there are reliable ways to deal with outliers, adding such modifications to the HLR architecture can improve its dependability.
iv. **Comparative Analysis with Emerging Techniques**: When the HLR estimator is compared with existing biased estimators, new machine learning-based methods that can provide alternative viewpoints on managing multicollinearity are not taken into account. Its respective benefits and drawbacks could be emphasized by a comparison study.

5.4 Directions for Future Research

i. Automated Parameter Selection: Create algorithms that pick *k* and *d* automatically and dynamically using data-driven techniques like Bayesian optimization and cross-validation to lessen dependency on preset values.
ii. Extending Application Domains: To assess the estimator's flexibility and scalability over a range of datasets and multicollinearity levels, test its application in a variety of fields, such as engineering, biology, and finance.
iii. Robustness to Outliers: Make adjustments to strengthen the HLR estimator's resistance to outliers, and use strategies like data preparation or robust regression to boost its effectiveness in practical settings.
iv. Incorporating Machine Learning Innovations: Examine hybrid models that provide a more comprehensive approach to managing multicollinearity in contemporary datasets by combining the HLR estimator with machine learning frameworks (such as LASSO and Elastic Net).

In addition to addressing the limitations, these approaches will increase the HLR estimator's applicability and relevance for a larger range of applications.

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Abbreviations

OLS	Ordinary Least Squares
MSE	Mean Square Error
RIDGE	Ridge Estimator
LIU	Liu Estimator
KL	Kibra-Lukman Estimator
NTP	New Two-Parameter Estimator
MRT	Modified Ridge Type Estimator
DK	Dawoud-Kibra Estimator
HLR	Hybrid Liu-Ridge Estimator

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