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A Comparative Study of MLE, MPS, WLS and Anderson–Darling Estimators for the Exponentiated Power Shanker Distribution.

Kenechukwu Florence Aforka*, Sidney I. Onyeagu and Harrison O. Etaga
Department of Statistics, Faculty of Physical Sciences Nnamdi Azikiwe University, Awka P.O. Box 5025, Nigeria
*Corresponding Author's E-mail: kf.aforka@unizik.edu.ng

Abstract
In this paper, we have considered different estimation methods of the unknown parameters of the Exponentiated Power Shanker (EPS) distribution. First, we briefly describe different methods of estimation such as maximum likelihood (MLE), Maximum Product of Spacings (MPS), Weighted Least Squares (WLS), methods of Anderson-Darling (AD), and compare them using extensive simulations studies. Finally, the potentiality of the model is studied using real data set related to the times between failures for repairable items which was initially studied by Murthy, Xie and Jiang (2004). These methods are compared using the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises (CVM), Anderson-Darling (A_n^2,) and Kolmogorov-Smirnov (KS) statistics. Results show that the MLE and AD are the best methods among the methods in the analytical and graphical procedures.

Keywords: Exponentiated Power Shanker Distribution, maximum likelihood estimation, Maximum Product of Spacings, Weighted Least Squares, methods of Anderson-Darling

1. Introduction

Lifetime and reliability data modeling is fundamental in biomedical sciences, engineering and reliability analysis. Classical distributions such as the exponent, Weibull, gamma and Lindley distributions are commonly used due to their simplicity and tractable properties. However, these models often lack the flexibility required to capture complex data characteristics such as heavy tails, skewness and non monotonic hazard rate functions frequently observed in real applications (Karakas and Bulut 2025; Warahena-Liyanage et al. 2023). To address these limitations, several generalized and extended distributions have been proposed in recent years. Among them, the Shanker type and power transformed distributions have gained increasing attention due to their improved flexibility and interpretability in lifetime modeling (Shanker 2015; Shanker and Shukla 2017). The Power Shanker distribution extends the Classical Shanker model by introducing an additional shape parameter, thereby enhancing its ability to model diverse hazard rate behaviours. Motivated by this development, the Exponentiated Power Shanker (EPS) distribution was recently introduced by Aforka et al. (2025), where the exponentiation parameter c was incorporated to further improve model flexibility and making it suitable for a wider range of reliability and survival datasets.

Although Aforka et al. (2025) introduces the EPS Distribution and investigated some of its statistical properties, their study mainly focused on model formulation and basic estimation using the Maximum Likelihood Estimator (MLE). A comprehensive evaluation of alternative classical estimation procedures and their finite sample performance for the EPS distribution has not yet been fully explored. In particular, there is limited empirical evidence regarding the relative performance of different estimators under varying sample sizes and parameter settings. Such investigations are important for guiding practitioners in selecting appropriate estimation methods for practical applications. Parameter estimation plays a crucial role in the practical application of any statistical model. The maximum likelihood estimation (MLE) method is the most widely used approach because of its desirable asymptotic properties including consistency and efficiency (Aforka et.al, 2025). Nevertheless, MLE may suffer from numerical instability, convergence issues and sensitivity to initial values, particularly for small sample sizes or complex likelihood surfaces (Maximum spacing estimation, n.d.). These Challenges have motivated the use of alternative classical estimation methods.

Among these alternatives, the maximum product of spacings (MPS) estimator, has been shown to be a robust and consistent alternative to MLE, especially in small samples and non-regular estimation problems. Techniques such as weighted least square (WLS) and the Anderson Darling (AD) estimation method provide additional options that emphasize distributional fit across different parts of the support with AD placing more emphasis on the tails of the distribution and has proven effective for skewed and heavy tailed data (Anderson and Darling test, n.d).
Together, a comparative study of these methods; MLE, MPS, WLS and AD can help identify the most appropriate strategies for parameter inference in flexible lifetime models such as the EPS distribution.

Despite the growing literature on exponentiated lifetime distributions, systematic comparisons of classical estimators for newly proposed models remain limited. In particular, the finite sample performance of MLE, MPS, WLS and AD estimators for the Exponentiated Power Shanker (EPS) distribution has not been thoroughly investigated. Such comparative studies are essential for guiding practitioners toward appropriate estimation methods under varying sample sizes and data structures. This study addresses this gap by providing a comprehensive comparison of four classical estimators for the EPS distribution. A Monte Carlo simulation study is conducted to assess estimator performance in terms of bias, mean squared error and convergence behavior. Furthermore, the applicability of the EPS model is illustrated using a real repair time dataset. The adequacy of the fitted model is assessed using graphical diagnostics and several goodness of fit measures including the Kolmogorov Smirnov (KS), Anderson Darling (AD) and Cramer von Mises (CVM) statistics, along with the information criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) and Hannan-Quinn Information Criterion (HQIC). The innovative aspect of this work lies in its systematic assessment of estimator performance across small and large samples, offering practical guidance for reliability analysts and expanding the understanding of EPS modelling in applied contexts.

2.0 Materials and methods

2.1 The Exponentiated Power Shanker distribution

The Exponentiated Power Shanker (EPS) distribution has been introduced by Aforka et al. [2025] as a distribution with a flexible lifetime model specifically designed for positive time to event data, particularly those that are skewed or right-censored.

2.1.1 Fundamental functions of the EPS distribution

$$X \sim \text{EPS}(c, \theta, \alpha), c > 0, \theta > 0, \alpha > 0$$

The cumulative distribution function (CDF) is given by

$$F(x; c, \theta, \alpha) = \left[1 - \left(1 + \frac{\theta x^{\alpha}}{\theta^2+1}\right) e^{-\theta x^{\alpha}}\right]^{c}, x > 0, c > 0, \theta > 0, \alpha > 0 \tag{1}$$

The probability density function (pdf) is obtained by differentiating the CDF and is given as:

$$f(x; c, \theta, \alpha) = \frac{c\alpha\theta^2}{\theta^2+1}(\theta + x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}\left[1 - \left(1 + \frac{\theta x^{\alpha}}{\theta^2+1}\right) e^{-\theta x^{\alpha}}\right]^{c-1} \tag{2}$$

Where c > 0 is the exponentiation parameter, $\theta > 0$ is the scale parameter and α > 0 is the shape parameter.
The survival function is given as:
$S(x; c, \theta, \alpha) = 1 - F(x; c, \theta, \alpha)$
which gives:

$$S(x; c, \theta, \alpha) = 1 - \left\{1 - \left[1 + \frac{\theta x^{\alpha}}{\theta^2+1}\right] e^{-\theta x^{\alpha}}\right\}^{c} \tag{3}$$

The hazard rate function of the EPS distribution is given by:

$$h(x; c, \theta, \alpha) = \frac{f(x; c, \theta, \alpha)}{1-F(x; c, \theta, \alpha)} \tag{4}$$

Further statistical properties of EPS distribution, including the moments (mean, variance, skewness and kurtosis), R´eny entropy, the moment generating function and the characteristic function has been discussed in Aforka et.al (2025).

2.1.2 Special cases and Model flexibility

The EPS distribution reduces to the Power Shanker distribution when c = 1
The parameter α controls tail thickness and skewness

The parameter c governs the overall shape and tail behavior
The parameter θ acts as a scale and rate parameter
By varying these parameters, the EPS distribution can accommodate light tailed and heavy tailed data as well as monotone and non-monotone hazard structures.

2.2 Parameter Estimation Methods

This section introduces and compares four estimation methods; the Maximum Likelihood Estimator (MLE), Maximum Product Spacings Estimator (MPS), Weighted Least Squares Estimator (WLS) and Anderson Darling Estimator (AD) for the EPS distribution.

2.2.1 Random Sample and Parameter Vector

Let $X_{1,} X_{2,} \dots X_n$ be a random sample of size n drawn from the exponentiated Power Shanker distribution with parameter vector

$$\Im = (c, \theta, \alpha), c > 0, \theta > 0, \alpha > 0 \quad (5)$$

With probability density function $f(x; \Im)$ and cumulative distribution function $F(x; \Im)$.

2.2.2 Maximum Likelihood Estimator (MLE)

The method of maximum likelihood is the most widely used estimation technique due to its desirable asymptotic properties such as consistency, efficiency and asymptotic normality. Given the EPS density function $f(x; \Im)$, the log-likelihood function based on a random sample $x_1, x_2, \dots, x_n$ is given by

$$l(\Im) = \sum_{i=1}^{n} log\ f(x_i; \Im) \quad (6)$$

The maximum likelihood estimates $(\hat{c}, \hat{\theta}, \hat{\alpha})$ are obtained by numerically maximizing $\ell(\Im)$ with respect to the parameters. Since no closed form solutions are available, numerical optimization techniques such as optim () in R with Nelder-Mead is employed.

2.2.3 Maximum Product Spacings Estimator (MPS)

The maximum product of spacings method is an alternative to MLE that often provides better finite sample performance and improved numerical stability, especially for complex distributions.
Let $X_{(1)} \leq X_{(2)} \leq \cdots X_{(n)}$ denote the order statistics, define the spacings

$$D_1 = F(X_{(1)}), \quad D_i = F(X_{(i)}) - F(X_{(i-1)}), i = 1,2, \dots, n, D_{n+1} = 1m - F(X_{(n)}). \quad (7)$$

The MPS estimator maximizes the geometric mean of these spacings, or equivalently minimizes the negative log-spacing function and is given as:

$$S(\Im) = -\sum_{i=1}^{n+1} logD_i \quad (8)$$

The MPS estimates

$$(\hat{c}_{MPS}, \ \hat{\theta}_{MPS}, \hat{\alpha}_{MPS})$$

are obtained by minimizing $S(\Im)$ numerically. The MPS method is known to be robust and consistent under mild regularity conditions and performs well when MLE encounters convergence issues.

2.2.4 Weighted Least Squares Estimation (WLS)

The weighted least squares estimator is based on minimizing the discrepancy between the theoretical cumulative distribution and the empirical distribution function. Let $X_{(i)}$ be the ith order statistic. The WLS estimator minimizes

$$W(\Im) = \sum_{i=1}^{n} w_i \left[F(X_{(i)}; \ \Im) - \frac{i}{n+1}\right]^2, \quad (9)$$

where the weights are defined as

$$w_i = \frac{(n+1)^2}{i(n-i+1)} \quad (10)$$

These weights assign greater importance to the tail observations. The WLS estimates
$(\hat{c}_{WLS}, \ \hat{\theta}_{WLS}, \ \hat{\alpha}_{WLS})$
are obtained by minimizing $W(\Im)$ numerically.

2.2.5 Anderson Darling Estimation (AD)

The Anderson Darling estimator is derived from the Anderson Darling goodness of fit statistic and emphasizes the tails of the distribution more strongly than ordinary least squares methods. The Anderson Darling objective function is given by

$$A(\Im) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF\left(X_{(i)};\Im\right) + \log\left(1 - \mathrm{F}\left(X_{(n+1-i)};\Im\right)\right)\right] \quad (11)$$

The AD estimates ($\hat{c}_{AD}$, $\hat{\theta}_{AD}$, $\hat{\alpha}_{AD}$) are obtained by minimizing $A(\Im)$. This estimator is particularly useful when accurate tail fitting is of primary interest.

2.2.6 Numerical Implementation

All the above mentioned estimation methods are implemented using numerical optimization routines. Due to the absence of closed form solutions, the Nelder-Mead algorithm is employed. Positivity constraints on the parameters are enforced during optimization. All simulations and numerical computations were carried out using R statistical software. Random samples from the EPS distribution were generated using the acceptance-rejection sampling algorithm based on a generalized exponential proposal distribution. Parameter estimation for the EPS distribution was obtained by minimizing the respective objective functions corresponding to the Maximum Likelihood Estimator (MLE), Maximum Product Spacings (MPS), Weighted Least Square (WLS) AND Anderson Darlington (AD) estimators. The optimization procedure was implemented using the optim () function in R with the Nelder Mead algorithm.

The maximum number of iteration for the numerical optimization was set to 8000 and convergence was assessed using the standard convergence criterion implemented in the optim () routine. Only estimates with convergence code equal to zero were retained for computing the Monte Carlo summaries. To ensure reproducibility of the simulation results, a fixed random seed (set.seed(123)) was used prior to the Monte Carlo experiments. For each parameter configuration, 1000 Monte Carlo replications were performed for sample sizes n = 20, 100, 300 and 600. The performance of the estimators was evaluated using the mean estimates, bias and Mean Square error (MSE).

3.0 Result and Discussion

3.1 Monte Carlo Simulation Study

3.1.1 Simulation Design

A Monte Carlo simulation study was conducted to assess the finite sample performance of the four estimators selected for the EPS distribution. The study evaluates the estimation accuracy, efficiency and numerical stability under different sample sizes.

3.1.1.1 True parameter Values

The simulation were conducted under using four different true parameter configurations each corresponding to a specific combination of (c , θ, α). Each scenario were chosen to represent flexible and realistic shapes of the EPS distribution and to allow the performance of the estimators to be evaluated under different distributional conditions.
The true parameter combinations considered in the simulation study were (c, θ, α) = (1.75, 2.5, 0.75), (1.25, 3, 2.25), (1.20, 2.5, 1.75) and (2, 1.5, 0.5). These values were selected to generate variety of distributional shapes, skewness levels and hazard rate behaviors of the EPS model. Such parameter settings are commonly used in Monte- Carlo studies to evaluate the robustness and stability of competing estimators under different model conditions. In particular, smaller values of the shape parameter (e.g α = 0.5) produce heavier tails, while larger values (e.g, α = 2.25) lead to lighter tails and different hazard vrate patterns. Similarly, varying the exponentiation parameter c and the scale parameter θ allows the distribution to exhibit diverse hazard shapes. Therefore, these parameter settings were selected to ensure that the simulation study evaluates estimator performance across a range of realistic and challenging scenarios for the EPS distribution.

3.1.1.2 Sample Sizes

The following sample sizes were used to evaluate the performance of the estimators:
n = 20, 100, 300, 600.
These sample sizes are standard and widely used in Monte Carlo studies and they allow the assessment of;
Small sample behavior (n = 20)
Moderate sample performance (n = 100)
Large sample and asymptotic properties (n = 300, 600).

This range enables assessment of how the estimators improve as sample size increases.

3.1.1.3 Number of Replications

For each sample size, the simulation was repeated 1000 times to ensure stable and accurate approximation of the performance measures. This number of replication is sufficient to reduce Monte Carlo error and is also consistent with the existing literature.

3.1.1.4 Performance Metrics

The estimators were evaluated using three commonly adopted performance measures, which are Bias, Mean Square Error (MSE) and Failure rate. These criteria collectively assess estimation accuracy, efficiency and numerical convergence behavior.

3.1.2 Performance Measures

Let φ denote the true parameter of the EPS distribution and $\hat{\varphi}^{(r)}$ be its estimate obtained from the r^{th} Monte Carlo replication, for $r = 1, 2, \dots., 1000$

3.1.2.1 Bias

Bias measures the systematic deviation of the estimator from the true parameter value. The bias of an estimator $\hat{\varphi}$ is defined as:

$$Bias\ (\hat{\varphi}) = \frac{1}{1000}\sum_{r=1}^{1000} \hat{\varphi}^{(r)} - \varphi \qquad (12)$$

3.1.2.2 Mean Square Error (MSE)

MSE reflects both the bias and variability of the estimator and it is a key measure of estimation efficiency.
The MSE for the simulation is defined as:

$$MSE\ (\hat{\varphi}) = \frac{1}{1000}\sum_{r=1}^{1000} (\hat{\varphi}^{(r)} - \varphi)^2 \qquad (13)$$

3.1.2.3 Failure Rate

The failure rate quantifies the proportion of simulation runs in which the estimation procedure fails to converge or yields invalid parameter estimates. It is defined as:

$$\text{Failure Rate} = \frac{Number\ 0f\ non\text{-}convergent\ replications}{1000} \times 100\% \qquad (14)$$

A lower failure rate indicates better numerical stability and robustness of the estimation method

3.1.3 Discussion of Simulation Results

This section discusses the finite sample performance of the estimators; MLE, MPS, WLS and AD for the EPS distribution based on the Monte Carlo simulation results summarized in tables 1-4.

Table 1: Simulation Results for true parameters $c = 1.75$, $\theta = 2.5$, $\alpha = 0.75$

n	Estimators	$\hat{c}$	$\hat{\theta}$	$\hat{\alpha}$	Bias(c)	MSE(c)	Bias(θ)	MSE(θ)	Bias(α)	MSE(α)	Failure (%)
20	MLE	1195057.6	3.041	1.249	1195055.8	5.14×10^{14}	0.541	5.194	0.499	2.160	1.7
	MPS	771761.1	2.895	1.013	771759.4	1.97×10^{14}	0.395	5.861	0.263	0.998	1.3
	WLS	3319947.8	3.421	1.191	3319946.0	1.44×10^{15}	0.921	18.495	0.441	2.029	2.7
	AD	1017652.5	3.158	1.028	1017650.7	3.94×10^{14}	0.658	6.436	0.278	0.901	1.2
100	MLE	2.157736	2.556	0.814	0.408	4.821	0.056	0.329	0.064	0.095	0
	MPS	2.137	2.503	0.779	0.387	4.634	0.003	0.304	0.029	0.070	0
	WLS	2.484	2.571	0.805	0.734	15.830	0.071	0.429	0.055	0.093	0
	AD	2.342	2.578	0.797	0.592	10.045	0.078	0.374	0.047	0.077	0
300	MLE	1.849	2.514	0.766	0.099	0.370	0.014	0.096	0.016	0.015	0
	MPS	1.853	2.497	0.751	0.103	0.365	-0.003	0.092	0.001	0.014	0
	WLS	1.918	2.524	0.762	0.168	0.628	0.024	0.127	0.012	0.021	0
	AD	1.909	2.525	0.762	0.159	0.570	0.025	0.120	0.012	0.019	0
600	MLE	1.793	2.508	0.759	0.043	0.150	0.008	0.043	0.009	0.007	0
	MPS	1.800	2.501	0.750	0.050	0.150	0.001	0.042	0.000	0.006	0
	WLS	1.820	2.513	0.757	0.070	0.216	0.013	0.055	0.007	0.009	0
	AD	1.821	2.515	0.757	0.071	0.210	0.015	0.054	0.007	0.008	0

Table 2: Simulation Results for true parameters $c = 1.25,\ \theta = 3,\ \alpha = 2.25$

n	**Estimators**	$\hat{c}$	$\hat{\theta}$	$\hat{\alpha}$	**Bias(c)**	**MSE(c)**	**Bias(θ)**	**MSE(θ)**	**Bias(α)**	**MSE(α)**	**Failure (%)**
20	MLE	390913.6	4.186	4.224	390912.3	5.2×10^{13}	1.186	17.508	1.974	29.492	3.8
	MPS	150004.4	3.837	3.419	150003.1	9.1×10^{12}	0.837	31.246	1.169	14.636	0.9
	WLS	4279943.3	6.331	3.553	4279942.1	2.5×10^{15}	3.331	1677.894	1.303	17.686	1.1
	AD	1376340.9	4.135	3.262	1376339.7	7.0×10^{14}	1.135	20.707	1.012	11.415	0.6
100	MLE	1.466	3.067	2.440	0.216	1.170	0.067	0.284	0.190	0.650	0
	MPS	1.464	2.966	2.318	0.214	1.123	-0.034	0.255	0.068	0.522	0
	WLS	1.640	3.070	2.420	0.390	2.462	0.070	0.349	0.170	0.871	0
	AD	1.596	3.082	2.393	0.346	2.015	0.082	0.323	0.143	0.714	0
300	MLE	1.282	3.005	2.320	0.032	0.156	0.005	0.080	0.070	0.137	0
	MPS	1.293	2.970	2.266	0.043	0.156	-0.031	0.077	0.016	0.122	0
	WLS	1.309	3.003	2.311	0.059	0.223	0.003	0.090	0.061	0.172	0
	AD	1.309	3.009	2.307	0.059	0.208	0.009	0.088	0.057	0.160	0
600	MLE	1.277	3.016	2.272	0.027	0.063	0.016	0.037	0.022	0.059	0
	MPS	1.288	2.998	2.238	0.038	0.063	-0.002	0.036	-0.012	0.055	0
	WLS	1.284	3.013	2.278	0.034	0.089	0.013	0.044	0.028	0.082	0
	AD	1.287	3.016	2.274	0.037	0.087	0.016	0.043	0.024	0.078	0

Table 3: Simulation Results for true parameters $c = 1.20,\ \theta = 2.5,\ \alpha = 1.75$

n	**Estimators**	$\hat{c}$	$\hat{\theta}$	$\hat{\alpha}$	**Bias(c)**	**MSE(c)**	**Bias(θ)**	**MSE(θ)**	**Bias(α)**	**MSE(α)**	**Failure (%)**
20	MLE	584874.6	2.970	3.282	584873.4	1.35×10^{14}	0.470	4.478	1.532	15.295	2.4
	MPS	257165.8	2.756	2.572	257164.6	3.36×10^{13}	0.256	6.703	0.822	6.712	0.07
	WLS	3850878.8	3.237	2.779	3850877.6	2.0×10^{15}	0.737	13.358	1.029	8.911	2.2
	AD	1476615.3	3.058	2.523	1476614.1	5.2×10^{14}	0.558	5.594	0.773	5.345	0.07
100	MLE	1.409	2.574	1.868	0.209	0.851	0.074	0.254	0.118	0.332	0
	MPS	1.410	2.512	1.773	0.210	0.830	0.012	0.225	0.023	0.257	0
	WLS	1.791	2.596	1.851	0.591	39.525	0.096	0.367	0.101	0.470	0
	AD	1.550	2.602	1.831	0.350	1.724	0.102	0.311	0.081	0.370	0
300	MLE	1.256	2.521	1.781	0.056	0.138	0.021	0.073	0.031	0.077	0
	MPS	1.267	2.503	1.738	0.067	0.136	0.003	0.069	-0.012	0.069	0
	WLS	1.292	2.525	1.776	0.092	0.227	0.025	0.092	0.026	0.112	0
	AD	1.294	2.530	1.772	0.094	0.220	0.030	0.091	0.022	0.105	0
600	MLE	1.230	2.512	1.763	0.030	0.056	0.012	0.035	0.013	0.036	0
	MPS	1.240	2.504	1.736	0.040	0.057	0.004	0.034	-0.014	0.034	0
	WLS	1.237	2.510	1.765	0.037	0.076	0.010	0.043	0.015	0.050	0
	AD	1.240	2.513	1.762	0.040	0.076	0.013	0.042	0.012	0.048	0

Table 4: Simulation Results for true parameters $c = 2$, $\theta = 1.5$, $\alpha = 0.5$

n	**Estimators**	$\hat{c}$	$\hat{\theta}$	$\hat{\alpha}$	**Bias(c)**	**MSE(c)**	**Bias(θ)**	**MSE(θ)**	**Bias(α)**	**MSE(α)**	**Failure (%)**
20	MLE	1716019.9	2.127	0.611	1716017.9	1.73×10^{15}	0.627	5.767	0.111	0.218	1.3
	MPS	140822.8	1.980	0.583	140820.8	6.90×10^{12}	0.480	4.857	0.083	0.197	1.6
	WLS	5683231.3	2.459	0.609	5683229.3	2.69×10^{15}	0.959	11.201	0.109	0.279	2.6
	AD	2725704.0	2.225	0.589	2725702.0	1.05×10^{15}	0.725	7.340	0.089	0.195	0.6
100	MLE	2.765685	1.558	0.517	0.766	72.603	0.058	0.239	0.017	0.014	0
	MPS	2.707745	1.540	0.503	0.708	79.127	0.040	0.235	0.003	0.013	0
	WLS	3.191814	1.608	0.510	1.192	47.338	0.108	0.377	0.010	0.021	0
	AD	2.881527	1.589	0.511	0.882	24.755	0.089	0.306	0.011	0.018	0
300	MLE	2.123242	1.513	0.506	0.123	0.463	0.013	0.059	6.41×10^{-3}	0.004	0
	MPS	2.103948	1.509	0.500	0.104	0.448	0.009	0.058	-8.87×10^{-5}	0.004	0
	WLS	2.218342	1.531	0.503	0.218	0.899	0.031	0.091	3.36×10^{-3}	0.006	0
	AD	2.212812	1.532	0.503	0.213	0.829	0.032	0.086	3.26×10^{3}	0.006	0
600	MLE	2.063372	1.512	0.502	0.063	0.170	0.012	0.027	0.002	0.002	0
	MPS	2.055612	1.510	0.498	0.056	0.168	0.010	0.027	-0.002	0.002	0
	WLS	2.101306	1.521	0.500	0.101	0.268	0.021	0.039	0.000	0.003	0
	AD	2.101116	1.521	0.500	0.101	0.258	0.021	0.037	0.000	0.003	0

Table 1 presents the simulation results for the EPS distribution under the true parameter values ($c = 1.75, \theta = 2.5, \alpha = 0.75$), using four estimation methods; MLE, MPS, WLS and AD for different sample sizes n= 20, 100, 300, 600. Estimator performance is assessed using Bias, Mean Square Error (MSE) and Failure rate. For small sample size n = 20, all estimators performed poorly, particularly the shape parameter c, the estimates $\hat{c}$ are extremely inflated (of order 10^6), resulting in very large bias, extremely large MSE (up to10^{15}), this indicates numerically instability and convergence difficulties in small samples. Among the estimators MPS and AD exhibit relatively smaller bias and MSE compared to MLE and WLS. The WLS performs worst, with the largest bias, MSE and the highest failure rate (2.7%). MLE also shows instability with a non-negligible failure rate (1.7%).

For moderate sample size (n =100), there is a dramatic improvement across all estimators, parameter estimates are now close to the true values, bias and MSE for all parameters decrease substantially and no convergence failures are observed. MPS and MLE yield the smallest bias and MSE for all parameters. WLS still exhibits noticeably larger MSE, particularly for c, AD performs competitively but slightly worse than MLE and MPS. For large sample (n = 300 and n = 600), all estimators produce estimates very close to the true parameter values, show negligible bias, have small MSE, exhibit zero failure rates. Bias and MSE decreases monotonically as sample size increases, confirming consistency of all estimators. MLE and MPS consistently yield the smallest MSEs, particularly for θ and α. WLS remains less efficient with relatively larger MSEs. AD performs well but does not outperform MLE or MPS. Similar patterns are observed in Table 2-4, corresponding to different combinations of the true parameter values. Across all scenarios, bias and MSE decrease as the sample size increases, confirming the consistency of all estimators. In all cases, the MLE and MPS consistently outperform the WLS and AD estimators in terms of efficiency. The relative ranking of the estimators remains unchanged across parameter configurations, indicating that the performance of the proposed estimation methods is robust to variations in the underlying parameter values and sample characteristics. A few extreme parameter estimates occurred in small sample simulations, which is expected for complex lifetime models, but these rare instances did not affect the overall results.

3.2 Real Data Application

3.2.1 Data Description

To illustrate the applicability and practical usefulness of the proposed Exponentiated Power Shanker (EPS) distribution, a real-life reliability data is analyzed. The data consist of the times between failures for repairable items which was initially studied by Murthy, Xie and Jiang (2004), as shown in Table 5.

Let $X_1, X_2, \ldots, X_n$ denote the observed failure times, assumed to be independently and identically distributed random variables following the EPS Distribution. The primary objective of this section is to assess the performance of the EPS model using different estimation techniques, namely the MLE, MPS, WLS and AD.

Table 5: Repair Times Data

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74
1.23	0.94	4.36	0.4	1.74	4.73	2.23	0.45	0.7	1.06
1.46	0.3	1.82	2.37	0.63	1.23	1.24	1.97	1.86	1.17

3.2.2 Parameter Estimation

The unknown parameters c, θ and α of the EPS distribution are estimated using four classical and robust estimation methods: MLE, MPS, WLS and AD. Each estimation method produces a distinct set of parameter estimates: Each estimation method produces a distinct set of parameter estimates:

$(\hat{c}, \hat{\theta}, \hat{\alpha})_{MLE}$, $(\hat{c}, \hat{\theta}, \hat{\alpha})_{MPS}$, $(\hat{c}, \hat{\theta}, \hat{\alpha})_{WLS}$, $(\hat{c}, \hat{\theta}, \hat{\alpha})_{AD}$

These estimates are subsequently used to evaluate the adequacy of the EPS distribution in modelling the observed failure times. Table 6 presents the estimates of the EPS distribution parameters obtained using MLE, MPS, WLS and AD methods.

Table 6: Parameter estimates of the EPS distribution for the real reliability data using MLE, MPS, WLS and AD methods

Estimator	$\hat{c}$	$\hat{\theta}$	$\hat{\alpha}$
MLE	2.26677	1.43577	0.91035
MPS	1.00000	1.00000	1.00000
WLS	3.00784	1.67361	0.77382
AD	2.80184	1.60530	0.83192

It is observed that the estimates vary slightly across methods, reflecting differences in the underlying optimization criteria. The MLE and AD methods yield relatively close estimates, indicating stability and consistency for this dataset. In contrast, the WLS method produces comparatively larger estimates of the shape parameter c suggesting higher sensitivity to tail behavior. Overall, the similarity of estimates across methods support the adequacy of the EPS distribution for modelling the given reliability data.

3.2.3 Graphical Assessment

Figure 1 compares the empirical distribution of the failure to time data with the fitted EPS density functions obtained using MLE, MPS, WLS and AD.

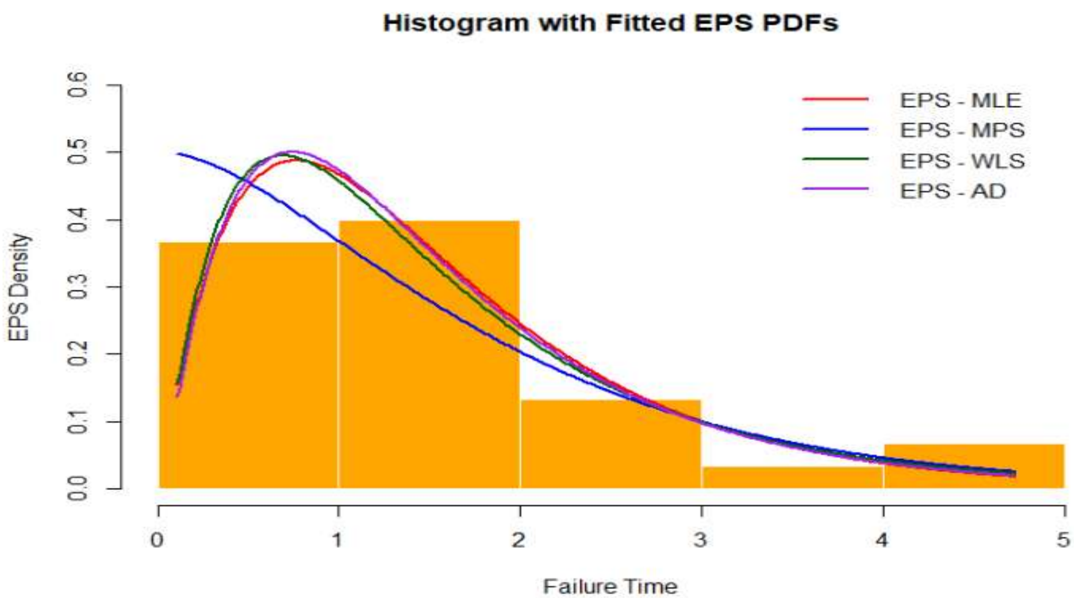

Figure 1: Histogram with fitted EPS PDFs under different Estimation Methods

The histogram exhibits a right skewed pattern, characterized by a high concentration of early failures and a long right tail, which is typical of reliability data. All four fitted EPS curves adequately reproduce the overall shape of the empirical distribution, confirming the flexibility of the EPS model for modelling failure to time behavior. Among the estimators, the MLE and AD methods provide closer agreement with the empirical density in both the central region and the right tail, indicating superior representation of typical and extreme failure times. MPS is competitive in the tail but less accurate at the peak, while the WLS yields reasonable fit but shows mild tail deviation. Overall, the graphical evidence favors MLE and AD as most accurate density fitting methods for the EPS model.

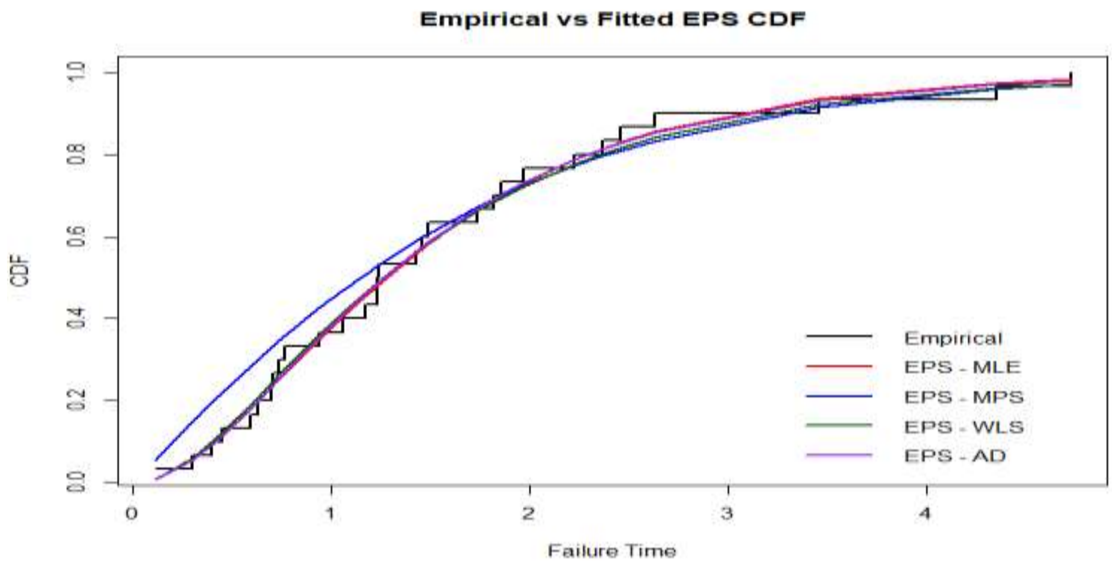

Figure 2: Empirical and Fitted Distribution Functions for the EPS model

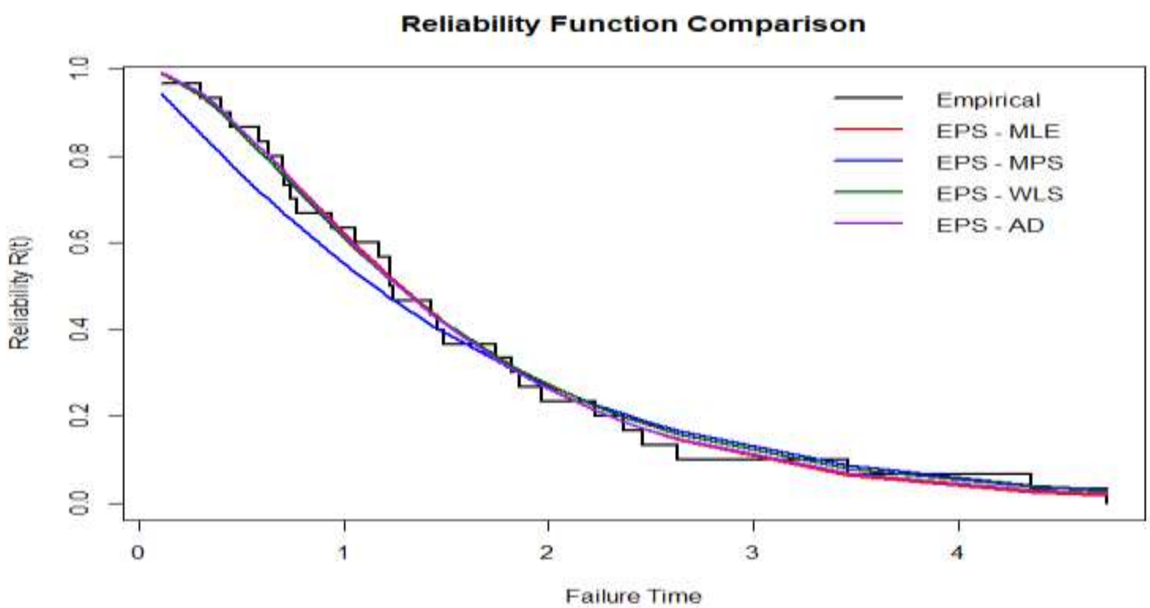

Figure 3: Reliability Function Comparison for the EPS model

Figure 2 assesses how well each estimator reproduces the overall distributional behavior of the real failure-time data. All four fitted EPS CDFs closely follow the empirical CDF (the black step curve), indicating that the EPS model provides an adequate fit to the data. The MLE (red line) and AD (purple line) curves track the empirical CDF more closely across most of the failure-time range, especially in the central region, where most observations lie. Slight deviations are visible at very small and large failure times, which is typical in finite samples and reflects sensitivity to tail behavior. Figure 3 focuses on the survival behavior, which is central in reliability analysis. All fitted EPS reliability curves (estimated using MLE, MPS, WLS and AD) exhibit a monotonically decreasing pattern, consistent with failure data. The fitted curves closely follow the empirical reliability function (the black step curve) throughout the time range, confirming that the EPS model captures the decay pattern of survival probability. The MLE and AD estimators provide curves that align more closely with the empirical function at early and intermediate failure times, where reliability inference is most critical. Minor discrepancies appear at later times due to fewer observations (right tail sparsity).

The graphical evidences presented in Figures 2 and 3 are consistent with the goodness of fit statistics reported in Table 7, where MLE and AD estimators yield smaller KS, A_n^2 and CVM values, confirming their superior performance for the real reliability data. Figure 4 presents the Q-Q plots of the EPS distribution fitted to the real reliability data using MLE, MPS, WLS and AD estimators

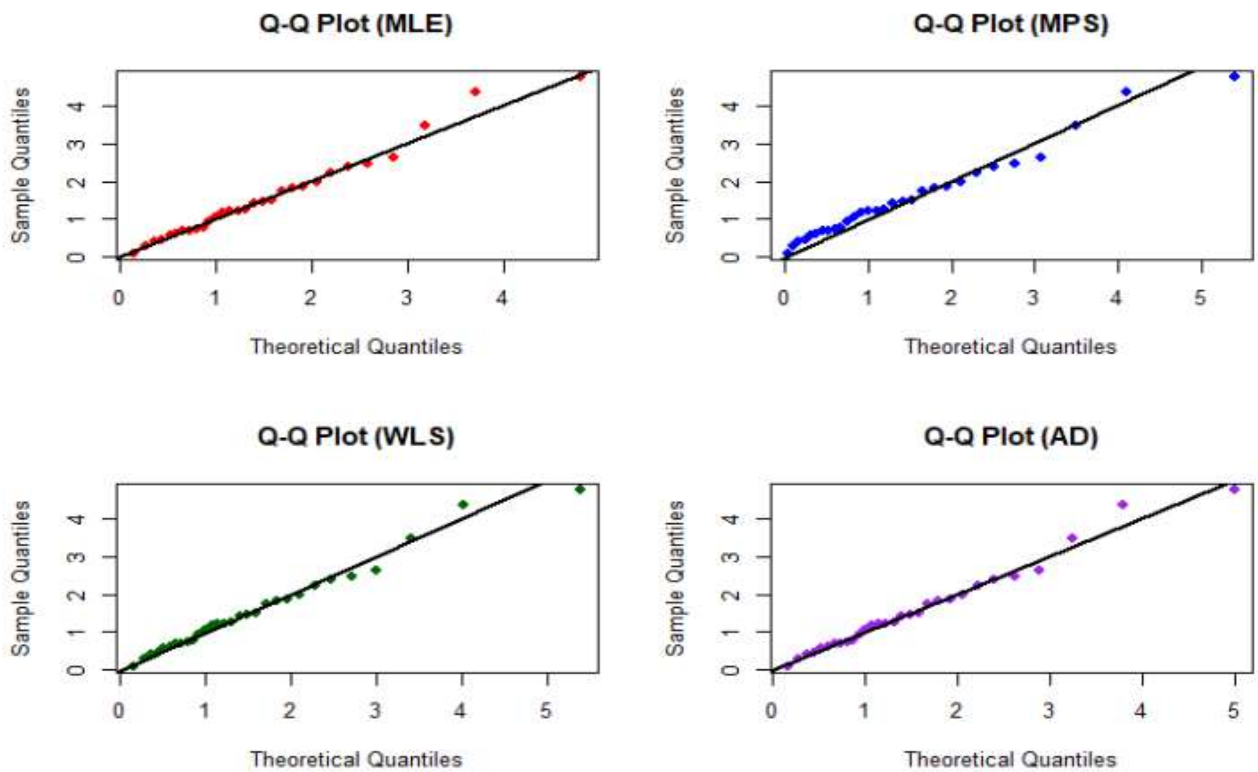

Figure 4: Q-Q plots of the EPS Distribution of the dataset under different Estimation Methods

Overall, the plots indicate that the EPS model provides an adequate representation of the data, as most points lie near the 450 reference line. The MLE and AD estimators show the strongest agreement with the empirical quantiles across the entire range including the tails, indicating superior global and tail fitting performance. The MPS displays more pronounced deviation from the reference line, indicating a comparatively weaker fit for the data. The WLS estimator shows noticeable departures from the reference line in the upper quantiles, while the lower and middle portions of the data align well, this also indicates a relatively weaker tail fit under WLS. These visual diagnostics are consistent with the goodness of fit statistics and information criteria, supporting MLE and AD as the most reliable estimators for the EPS model on the real dataset.

Figure 5 presents the P-P Plots that compare the empirical cumulative distribution function (CDF) of the real dataset with the theoretical CDFs obtained from MLE (red line), MPS (blue line), WLS (green line) and AD (purple line).

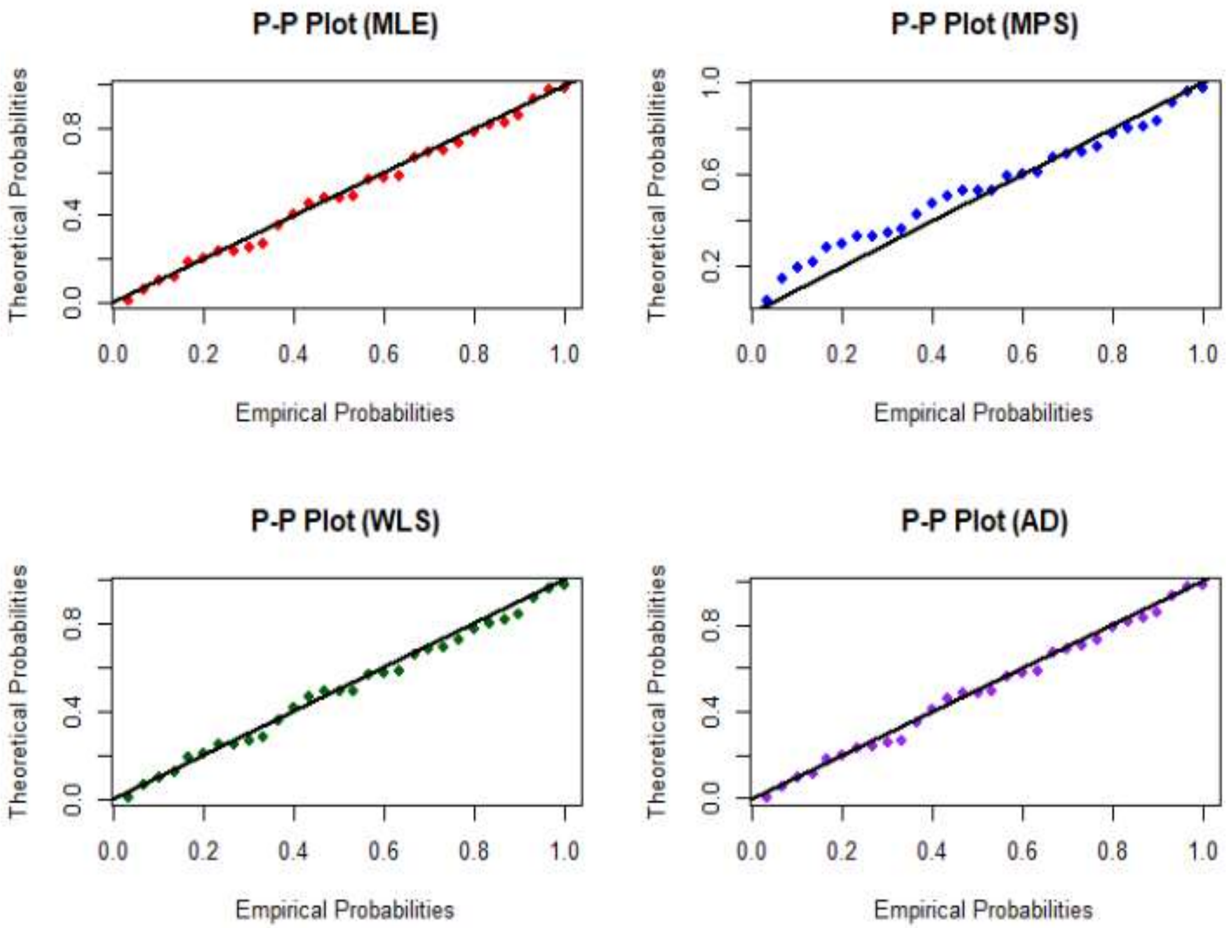

Figure 5: Probability-Probability (P-P) plots for the fitted EPS Distribution of the dataset under different Estimation Methods

The X-axis represents the empirical probabilities, while the Y-axis represents the corresponding theoretical probabilities based on the fitted parameters. The 450 reference line indicates perfect agreement between the observed and theoretical distributions. MLE provides the best overall fit because it closely followed the reference line across most of the probability range. WLS and AD also show reasonable fits with minor deviations in the tails. In contrast, MPS shows substantial deviation, particularly at lower probabilities, reflecting degenerate parameter estimates and poor convergence for the dataset. These plots visually confirm the numerical parameter estimation results and highlight the importance of robust estimation procedures in accurately capturing the underlying data distribution.

3.2.4 Goodness of Fit Comparison

To compare the performance of the four estimators on real data, several goodness of fit criteria are considered. These include both empirical distribution based tests and information theoretic criteria, ensuring a comprehensive assessment. The estimator selection criteria are the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) which are computed using;

AIC = $-2\ell(\widehat{\Im}) + 2k$,

CAIC = $-2\ell(\widehat{\mathfrak{I}}) + \frac{2kn}{(n-k-1)}$,

BIC = $-2\ell(\widehat{\mathfrak{I}})$ + k log (n),

HQIC = $-2\ell(\widehat{\mathfrak{I}})$ + 2k log (n),

Where $\ell(\widehat{\mathfrak{I}})$ is the log-likelihood function, k is the number if parameters, n represents the sample size and $\widehat{\mathfrak{I}}$ is the vector of parameters for the EPS model i.e $\widehat{\mathfrak{I}} = (\hat{c}, \hat{\theta}, \hat{\alpha})$. For goodness of fit test, let $X_{(1),}\ X_{(2),}\ ...\ X_{(n),}$ be an ordered random sample from the EPS (c, θ, α), where c, θ and α are unknown. The Cramer-von Mises (CVM), Anderson-Darling A_n^2 and Kolmogorov-Smirnov (KS) statistics are computed as follows;

$$CVM = \frac{1}{12n} + \sum_{j=1}^{n}[F(x_{j,}\ \hat{c}, \hat{\theta},, \hat{\alpha}) - \frac{2j-1}{n}]^2,$$

$$A_n^2 = -n - \sum_{j=1}^{n} \frac{2j-1}{n} [\ln\{F(x_{j,}\ \hat{c}, \hat{\theta},, \hat{\alpha})\} + \ln\{1 - F(x_{j,}\ \hat{c}, \hat{\theta},, \hat{\alpha})\}]$$

$$KS = max_j \{\frac{j}{n} - F(x_{j,}\ \hat{c}, \hat{\theta},, \hat{\alpha}), F(x_{j,}\ \hat{c}, \hat{\theta},, \hat{\alpha}) - \frac{j-1}{n}\}.$$

Estimators with smaller values of AIC, BIC, CAIA and HQIC are considered to provide a better fit to the data

Table 7: Goodness of Fit and performance Criteria for the Real Data using the EPS Distribution

Estimators	KS	A_n^2	CVM	Loglik	AIC	BIC	HQIC	CAIC
MLE	0.06650	0.12830	0.01658	-39.61482	85.22965	89.43324	86.57441	92.43324
MPS	0.14881	0.80601	0.13170	-41.56262	89.12523	93.32883	90.47000	96.32883
WLS	0.06235	0.14096	0.01980	-39.73502	85.47004	89.67363	86.81480	92.67363
AD	0.06424	0.12490	0.01651	-39.64068	85.28136	89.48495	86.62612	92.48495

From Table 7, the WLS estimator has the smallest KS value (0.06235), closely followed by AD (0.06424) and MLE (0.06650). MPS has the largest KS (0.14881), indicating it deviates more from the empirical distribution. The AD estimator itself gives the lowest A_n^2 of 0.1249, meaning it fits the tails of the distribution better. MPS again performs worst (0.80601), suggesting poor fit in the tails. AD and MLE have almost the same smallest CVM of 0.01651 and 0.01658 respectively, indicating the best overall fit. MPS again is the largest CVM of 0.13170, confirming poor overall fit.

Also table 7 presents the log-likelihood and information criteria values for the EPS distribution fitted to real reliability data using four estimation methods. Since all estimators involve the same number of parameters, smaller values of AIC, BIC, HQIC and CAIC indicate superior estimator performance. The results show that the MLE yields the highest log-likelihood and the minimum values across all information criteria, indicating the best overall fit. The WLS estimator performs competitively and ranks second, followed closely by the AD estimator. The MPS estimator has the largest values of all criteria, suggesting comparatively poorer performance for the dataset. Overall the ranking MLE > AD > WLS > MPS demonstrates the efficiency and robustness of MLE for modelling real reliability data under the EPS distribution. In Summary, Goodness of fit statistics for the four estimators indicate that the AD and MLE estimators provide the most accurate representation of the empirical reliability data, as evidenced by their consistently lower KS, A_n^2 and CVM values. In contrast, the MPS estimator shows the poorest fit, particularly in the distribution tails.

4.0. Conclusion

This study investigated the performance of different estimation methods for determining the parameters of the EPS Distribution. The primary objective was to provide a systematic comparison of four classical estimators; MLE, MPS, WLS and AD. Using Monte-Carlo Simulations and a real repair time dataset. The simulation results indicate that all estimators are consistent, with bias and mean squared error decreasing as sample size increases. Similar downward trends in MSE with increasing sample sizes have been observed in recent simulation studies involving various lifetime distributions and estimation techniques. For example, simulation results for several univariate distribution show that estimator performance generally improves with larger sample sizes with some estimators (like MPS) showing demonstrably lower bias and MSE under specific conditions (Ranade et al, 2026).

Consistent with comparative simulation research on new reliability estimators and classical methods, MPS and AD demonstrated superior finite sample performance and greater numerical stability than WLS especially in small samples. In a recent study on flexible laplace-gamma compound model, the MPS method consistently ranked highest in simulation ranking tables due to its robustness and efficiency, confirming that CDF based estimation criteria can outperform classical likelihood methods under complex sampling scenarios. (Ranade et al, 2026). Meanwhile, WLS exhibited higher variability, particularly in small samples, highlighting that least squares approaches may struggle with instability when modelling non-linear or non standard distributions, a trend similarly reported in simulation studies comparing regression based methods and likelihood based estimators (Elbatal et al, 2024).

The real data application further confirms the flexibility of the EPS distribution, as graphical diagnostics and goodness of fit measures (KS, A_n^2, CVM, AIC, BIC, HQIC, CAIC) demonstrate satisfactory fit. Among the estimators, MLE and AD provided the best overall performance, especially in capturing tail behavior, while MPS performed competitively in modelling the central region. This detailed comparison demonstrates that estimator choice should be guided by both sample size and the specific features of the data distribution, in agreement with recent recommendations from studies on parameter estimation techniques for new lifetime models (Hassan et al, 2023).

In conclusion, this study provides a comprehensive comparison of four classical estimation methods for the EPS distribution and highlights the difference in their performance under varying sample conditions. The results emphasize that while all estimators exhibit consistency. MLE and AD generally provide more reliable overall estimates with MPS also showing competitive performance in several cases. These findings contribute to the growing literature on parameter estimation for flexible lifetime distributions by offering empirical evidence on the relative performance of commonly used estimators. The results also provide practical guidance for researchers and practitioners when selecting estimation methods for EPS distribution in reliability and lifetime data analysis. Future research could extend this work by examining the performance of these estimators under censored or truncated data settings, exploring Bayesian or robust estimation approaches or applying the EPS distribution within regression-based reliability models that incorporate covariates.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author(s) used AI- assisted tools (e.g ChatGPT) in order to improve language clarity and formatting. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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