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Microalgae Lipid Accumulation: Dynamic Modelling and Optimal Control for Biojet Fuel Production

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Abstract

The global aviation sector is a major driver of climate change, currently generating roughly 2.5% of the world's carbon emissions—a figure that could easily triple by 2050 if left unchecked. To combat this, the International Air Transport Association (IATA) has set aggressive decarbonization goals, pushing for carbon-neutral growth and a 50% cut in net emissions by 2050 compared to 2005 benchmarks. Sustainable Aviation Fuel (SAF) stands out as the most practical, immediate solution to reduce the industry's carbon footprint, offering up to an 80% reduction in lifecycle emissions compared to traditional jet fuel. To unlock this potential, this study bridges the gap between raw experimental data and computational insights by developing a novel Python-based mathematical model to analyse microalgal lipid production. The framework successfully maps out the intricacies of algae population growth, lipid accumulation, and harvesting mechanics. Through this model, we established critical equilibrium conditions and calculated the basic lipid reproduction number (R_0). Our simulations validated the model, proving that the Algae-Lipid Free Equilibrium (ALFE) remains locally asymptotically stable under specific parameters. Furthermore, a sensitivity index analysis was conducted to determine which variables most heavily influence R_0. Ultimately, this research delivers a robust, validated computational framework that confirms the practical viability of community-scale, microalgae-based SAF while providing a clear roadmap for production optimisation.

Keywords: algae; harvesting; sustainable aviation fuel; biofuel; sensitivity analysis

1.0 Introduction

The study talks about the aviation sector's CO_2 emissions (~2.5% of anthropogenic total) and the expected rise by 2050, in line with IATA's carbon-neutral growth and 50% net emission reduction targets (IATA,2023). Sustainable aviation fuel (SAF), mostly from microalgae, is emphasised for its up to 80% lifecycle emission savings (Cui & Chen, 2024; Bell, 2025), voluminous oil yields (5,000–15,000 L/ha/year), and non-competition with food crops (Saratale, 2022). Africa, with annual fuel demand growth of ~4.7% through 2040, offers positive environments for algae cultivation (sunlight, non-arable land, wastewater) (Aidam, 2024; Bardon 2025), yet SAF development is inhibited by limited refining, policy gaps, and high capital costs (Echendu et al., 2023; Miftah & Mutta, 2024). Nigeria, as Africa's major economy, imports ~90% of its 500 million L/year jet fuel at ~$1.2 billion annually (Siddig, 2014; Adekitan, 2022; Laska & Ige, 2023), with no domestic SAF production (Ojo & Ayanwale, 2019; Oluwaseun, 2023), despite strategic policies recognising microalgae as a feedstock due to tropical climate, waste resources, and indigenous strains with 20–50% lipid content (Zhu, et al.,2022; Khoo et al., 2023). The study's main objective is to develop a computational model of algae lipid generation, growth, harvesting, and conversion to SAF; specific goals include formulating dynamic population and lipid accumulation models, establishing equilibrium conditions, computing the basic reproduction number (R_0), and performing Python numerical simulations for validation**.**

2.0 Materials and methods

This theoretical study modelled algae lipid production for sustainable aviation fuel and completed no lab work. The study used: Published literature to create a biological background and define parameters; Creating mathematical ordinary differential equation, ODE models and use stability and sensitivity analyses; Using Python and Maple software for numerical and symbolic calculations; and Assessing the lipid quality of the algae strains using published FAME profiles and determining the suitability for fuel based on their saturated fatty acids/monounsaturated fatty acids/polyunsaturated fatty acids, SFA/MUFA/PUFA. All parameters were taken from the literature or were estimated to fall within the reasonable biological ranges. The production dynamics and assessment of fuel potential were done using a combination of modelling, simulation, and interpretation of the literature.

2.1 Mathematical Modelling Framework

This study created a model to estimate lipid production by algae to produce sustainable aviation fuel. From a theoretical and computational perspective, the researchers created a five-compartment model (growing, stressed, high-lipid, biofuel, and genetically-modified algae). Each compartment's components were constructed using parameters from the literature and biological assumptions. Some relevant biological processes were included in the model, such as harvesting and conversion. The model also excluded stochastic elements and details of the reactors to ensure the model could be analysed. The output of the model was assessed for fuel suitability using a framework based on the literature

2.1.1 Model Formulation and Assumptions

A compartmental model was formulated by adapting the SIR (Susceptible-Infected-Recovered) epidemiological framework to algae lipid production dynamics. The model is based on the following assumptions: A homogeneous population of Chlorella vulgaris; Algae mortality occurs from both natural causes and environmental stressors; Lipid accumulation is dependent on nutrient availability and environmental exposure; Transplanted and genetically modified algae can be introduced into the system.

2.1.2 Model Flow Diagram

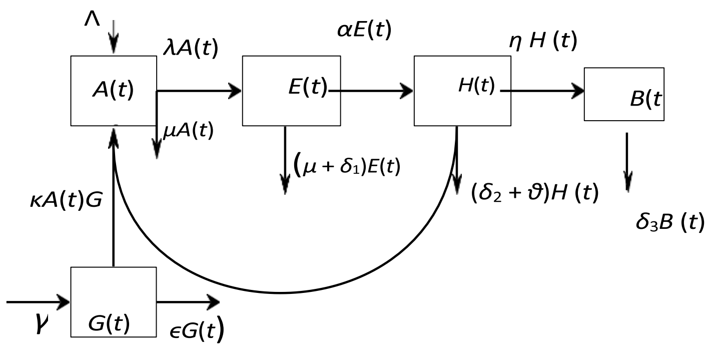

Figure 1: Flow diagram

2.1.3 Model Equations

Using the model assumptions together with the flow diagram, we formulated the governing system of equations for the proposed model as follows:

$$\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A \qquad (1)$$

$$\frac{dE}{dt} = \lambda A - [\alpha + \delta_1 + \mu]E \qquad (2)$$

$$\frac{dH}{dt} = \alpha E - [\delta_2 + \theta + \eta]H\ s \qquad (3)$$

$$\frac{dB}{dt} = \eta H - \delta_3 B \quad (4)$$

$$\frac{dG}{dt} = \gamma G - [\epsilon + k + \mu]G \quad (5)$$

The forces of algae lipid production is given as:

$$\lambda = \beta\left(1 - \frac{n(t)}{n_1}\right) E(t)L_m \quad (6)$$

Table 1. model Variables with description

Variables	Description
A	The population of algae capable of growth and reproduction
E	The population of algae exposed to environmental stressors that trigger lipid accumulation
H	Population of lipid-rich algae, ready for harvesting.
B	Population of algae converted into biofuel
G	Population of genetically grown algae.

Table 2. model parameters with description

Parameters	Description
Λ	Algae purchased or transplanted proportion
k	Rate of acceptance of genetically grown algae.
μ	Natural algae death rate
α	Lipid accumulation rate due to exposure
δ_1	Algae death rate due to environmental factors
δ_2	Harvest rate
λ	Lipid accumulation rate
η	Rate of conversion of algae lipid to biofuel
δ_3	Biofuel conversion rate
ϵ	Induced environmental exposure
θ	Spore formation rate
β	Algae contact rate
γ	Proportion of newly introduced genetically grown algae

3.0 Results and Discussion

3.1 Analysis and Discussion of Results

After deriving the model equations, we analyzed their mathematical properties to determine whether the formulation provides a biologically meaningful representation of algae lipid production dynamics. We first examine boundedness and continuity of the state variables within the feasible region.

3.1.1 Invariant Region: This subsection identifies a region of the state space in which the key properties of the dynamical system remain unchanged over time. Establishing such an invariant region is important because it clarifies the long-term behaviour of the model and shows where biologically meaningful trajectories are preserved.

3.1.1.1 Invariant Region Solution

From equations (1) to (5), which describe the overall algae population, it is clear that the state variables cannot take negative values. We therefore define the invariant Ω as follows:

$$\Omega = \{(A, E, H, B, G) \in R_+^5\} \quad (7)$$

Then

$$\{(A, E, H, B, G) \text{ and } N(t)\} \geq 0 \quad \forall\, t \geq 0 \quad (8)$$

In this study, the classical compartmental framework introduced by Kermack and McKendrick in 1927 was adapted and partitioned into the following subpopulations:

$$N = A + E + H + B + G \quad (9)$$

Theorem1: Suppose that equations (1) to (5) have solutions that are possible for all time $t \geq 0$, as long as they remain in the invariant region, $\Omega \in R_{+}^{5}$, where

$$\Omega = \{(A, E, H, B, G) \in R_{+}^{5}\} \geq 0 \tag{10}$$

Proof

We now show that the solutions of the algae lipid production model remain positive, following standard arguments commonly used in biological systems modelling (Mbah et al., 2023).

Assume that all initial conditions for the algae lipid production (ALP) model are positive. Under this assumption,
$A(0) > 0, E(0) > 0, H(0) > 0, B(0) > 0, G(0) > 0$
$\Rightarrow \{(A, E, H, B, G) \text{ and } N(t)\} \geq 0, \forall t \geq 0$

We show that Ω is positively invariant, so that it is sufficient to consider the dynamics of the model system.
From equation (9), which represents the algae population, we obtain the following differential equation:

$$\frac{dN}{dt} = \Lambda - \mu A - \mu E - \mu H - \mu B - \mu G - \delta_1 E - \delta_2 H - \delta_3 B \tag{11}$$

$$\frac{dN}{dt} = \Lambda - \mu N - \delta_1 E - \delta_2 H - \delta_3 B$$

$$\frac{dN}{dt} + \mu N = \Lambda - \delta_1 E - \delta_2 H - \delta_3 B \tag{12}$$

Equation (12) is then solved using the integrating factor method.
Equation (12) can be written in the standard first-order linear form:

$$y' + Py = Q \tag{13}$$

Its general solution is given by:

$$ye^{\int P dx} = \left[\int Q e^{\int P dx} dx + c\right],$$

Where $IF = e^{\int P dx}$

From equation (13), we identify the following quantities:

$$y' = \frac{dN}{dt}, y = N(t), P = \mu, Q = \Lambda - \delta_1 E - \delta_2 H - \delta_3 B, IF = e^{\int \mu dt} = e^{\mu t}$$

Now multiply (13) by $IF = e^{\mu t}$

$$e^{\mu t}\frac{dN}{dt} + e^{\mu t}\mu N = e^{\mu t}(\Lambda - \delta_1 E - \delta_2 H - \delta_3 B) \tag{14}$$

$$\Rightarrow \frac{d}{dt}(e^{\mu t} N) = e^{\mu t}(\Lambda - \delta_1 E - \delta_2 H - \delta_3 B)$$

Integrating both sides with respect to t gives:

$$\int \left(\frac{d}{dt}(e^{\mu t} N)\right) dt = \int \left(e^{\mu t}(\Lambda - \delta_1 E - \delta_2 H - \delta_3 B)\right) dt$$

$$e^{\mu t} N = \frac{e^{\mu t}(\Lambda - \delta_1 E - \delta_2 H - \delta_3 B)}{\mu} + C_1(t) \tag{15}$$

In the absence of lipid production, $E = H = B = 0$, then (15) becomes

$$\Rightarrow e^{\mu t} N(t) \leq \frac{e^{\mu t}\Lambda}{\mu} + C_1 \tag{16}$$

Divide (17) by $e^{\mu t}$

$$\Rightarrow e^{\mu t} N(t) \leq \frac{e^{\mu t}\Lambda}{\mu} + C_1$$

$$\therefore \quad N(t) \leq \frac{\Lambda}{\mu} + e^{-\mu t} C_1 \tag{17}$$

Now at $t = 0$

$$C_1 \geq N(0) - \frac{\Lambda}{\mu} \tag{18}$$

Substituting (19) into (18) gives:

$$N(t) \leq \frac{\Lambda}{\mu} + e^{-\mu t}\left(N(0) - \frac{\Lambda}{\mu}\right)$$

$$N(t) \leq N(0) e^{-\mu t} + \frac{\Lambda}{\mu}[1 - e^{-\mu t}] \tag{19}$$

$$\Rightarrow N(t) \leq N(0) e^{-\mu t} + \frac{\Lambda}{\mu}[1 - e^{-\mu t}] \tag{20}$$

Therefore, taking the limit of (21) as $t \rightarrow \infty$, to give

$$N(t) \leq \frac{\Lambda}{\mu} \tag{21}$$

$$\Rightarrow 0 \leq N(t) \leq \frac{\Lambda}{\mu} \tag{22}$$

From (22), it shows that the solutions for the algae population would always remain in Ω as long as $t \geq 0$. And the solution shows it is bounded below by 0 and above by $\frac{\Lambda}{\mu}$

Therefore, from (22), it was obtained that the solutions of (1) to (5) remains in Ω for all time, $t \geq 0$

3.1.2 Positivity of Solution

A central requirement in biological modelling is that solutions starting from positive initial conditions should remain positive for all future time (Psarros et al., 2017).

For the model in equations (1) to (5) to be biologically meaningful, every state variable must remain non-negative for all t. In other words, if the model starts with positive initial data, then the corresponding solutions must stay positive throughout the simulation period.

Because negative time has no physical interpretation in this context, the analysis is restricted to $t \geq 0$. If positivity fails, then the model loses biological validity.

$$\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A$$

$$\frac{dE}{dt} = \lambda A - [\alpha + \delta_1 + \mu]E$$

$$\frac{dH}{dt} = \alpha E - [\delta_2 + \theta + \eta]H$$

$$\frac{dB}{dt} = \eta H - \delta_3 B$$

$$\frac{dG}{dt} = \gamma G - [\epsilon + k + \mu]G$$

Theorem 2:

The solutions set for the general algae lipid production (ALP) model, are given by $\{ (A, E, H, B, G) \in \mathcal{R}_+^5 \}$, such that these components (A, E, H, B, G) must be present at the beginning of the ALP for the lipid production to have an effect in any situation/society. That is $\{ A(t), E(t), H(t), B(t), G(t) \} \geq 0$

Proof:

Let then the solution $\{ A(t), E(t), H(t), B(t), G(t) \text{ } and \text{ } N(t)\} \geq 0, \forall \, t > 0$ We show that the following algae feasible region (Ω) of the ALP model is positively invariant so that it is sufficient to consider the dynamics of the model system

From (1): $\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A$

$$\frac{dA}{dt} \geq -\left(\beta\left(1 - \frac{n}{n_1}\right) EL_m\right)A - \mu A \tag{23}$$

Recall: $\lambda = \beta\left(1 - \frac{n(t)}{n_1}\right) E(t)L_m$, substitute this into (24), to give

$$\frac{dA}{dt} \geq -\left(\beta EL_m\left(\frac{n_1 - n}{n_1}\right) + \mu\right)A \tag{24}$$

Then $\frac{dA}{A} \geq -\left(\beta EL_m\left(\frac{n_1 - n}{n_1}\right) + \mu\right) dt$ (25)

Integrating both sides of (26) yields: $\int \frac{dA}{A} \geq -\int \left(\beta EL_m\left(\frac{n_1 - n}{n_1}\right) + \mu\right) dt$

$$\Rightarrow \ln A \geq -\left[\beta EL_m\left(\frac{n_1 - n}{n_1}\right) + \mu\right]t + C_2 \tag{26}$$

Take exponential of both sides $\quad A(t) \geq e^{-\left(\left[\beta E L_m\left(\frac{n_1-n}{n_1}\right)+\mu\right]t+C_2\right)}$

At t=0 $\quad A(0) \geq C_2$

This implies that $\quad A(t) \geq 0 \quad (27)$

At all $t \geq 0$, the solution is positive and bounded.
Applying the same procedure to the remaining model equations shows that their solutions also remain positive for all time, $t \geq 0$.
Having established positivity, we next examine the existence and non-existence of lipid within the algae population by determining the equilibrium points corresponding to the lipid-free and lipid-abundant states.

3.1.3 Establishing the algae-lipid-free equilibrium (ALFE) points for the General Model

This subsection determines the equilibrium values of the parameters in equations (1) to (5) that describe a state in which no algae lipids are being produced, that is, a state in which none of the lipid-related compartments are active.
To obtain the equilibrium point, equations (1), (2), (3), (4), and (5) are set equal to zero, giving:

$$\frac{dA}{dt}=0, \quad \frac{dE}{dt}=0, \quad \frac{dH}{dt}=0, \quad \frac{dB}{dt}=0, \frac{dG}{dt}=0$$

$$0 = \Lambda + kG + \theta H - \lambda A - \mu A \quad (28)$$

$$0 = \lambda A - [\varphi + \alpha + \delta_1 + \mu]E \quad (29)$$

$$0 = \alpha E - [\delta_2 + \theta + \eta]H \quad (30)$$

$$0 = \eta H - \delta_3 B \quad (31)$$

$$0 = \varphi E + \gamma G - [\epsilon + k + \mu]G \quad (32)$$

Recall: $\lambda = \beta \left(\frac{n_1 - n}{n_1}\right) L_m E$

Solving equations (28) to (32) gives the algae-lipid-free equilibrium (ALFE) state for the ALP model as follows:

$$T^0 = (A^0, E^0, H^0, B^0, G^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right) \quad (33)$$

After obtaining the equilibrium point, the next task is to determine whether it is stable. To do this, we first derive the basic lipid reproduction number and then examine the growth conditions and stability properties of the lipid-free and lipid-abundant states.

3.1.4 The Basic Lipid Reproduction Number (R_0) for the Study

This section derives a key threshold parameter for assessing the likelihood of algae lipid production within an algae farm. The basic reproduction number represents the expected number of secondary lipid-generating units arising from an exposed algae population (Yang & Xu, 2019). It is one of the most important indicators for deciding whether lipid production can persist or die out over time.

To construct the basic reproduction number, we need to form the next generation matrix, which is FV^{-1}, where V is the matrix containing all of the transferred words and F is the newly exposed term. Additionally, F is the rate at which a newly exposed algae appears, and V is the difference between the rate at which lipid enter the same compartment by other means and the rate at which lipid exit the compartment by other means.

Reproduction number $R_0 = \rho(FV^{-1})$, where ρ denotes the spectral radius of a matrix FV^{-1}which represents the dominant non-negative eigenvalue of the next generation matrix of the square matrices F and V^{-1} of order (m x m) where m is the number of exposed class and defined by

$F = \frac{\delta fi}{\delta xj}(x_o)$ and $V = \frac{\delta vi}{\delta xj}(x_o)$

Such that f is non-negative, V is a nonsingular m-matrix, and x_o is the algae-lipid-free equilibrium point (ALFE). If $R_0 > 1$, it means that an exposed algae produces enough lipid throughout its lipid production period in the population and as such the algae lipid production increases through the population but if $R_0 < 1$, it means that an exposed algae produces on the average less lipid than is expected for an exposed algae which by calculation means that the lipid production will die out from the algae population with time .

Rewriting the general ALP model so that the lipid-related compartments appear explicitly, we obtain:

$$\frac{dE}{dt} = \lambda A - [\varphi + \alpha + \delta_1 + \mu]E$$
$$\frac{dH}{dt} = \alpha E - [\delta_2 + \theta + \eta]H$$
$$\frac{dB}{dt} = \eta H - \delta_3 B$$
$$\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A$$
$$\frac{dG}{dt} = \varphi E + \gamma G - [\epsilon + k + \mu]G$$

Recall: $\lambda = \beta \left(\frac{n_1 - n}{n_1}\right) L_m E$

where λ is the force of algae lipid production in the population.
From the model formulation, we define:
X = vector of the lipid production classes, such as exposed, lipid-carrier, etc.
Y = vector of non-lipid production classes, such as unexposed algae, unexposed genetically modified algae, etc.
Therefore, $X = \begin{pmatrix} E \\ H \\ B \end{pmatrix}$, $Y = \begin{pmatrix} A \\ G \end{pmatrix}$

$$f_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad v_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Where $f_i's$ are new lipid production terms, while $v_i's$ are transferred lipid production terms to other classes

$$f_i = \begin{bmatrix} f_1 = \beta \left(\frac{n_1 - n}{n_1}\right) L_m E A \\ f_2 = 0 \\ f_3 = 0 \end{bmatrix} \tag{34}$$

$$v_i = \begin{bmatrix} v_1 = [\varphi + \alpha + \delta_1 + \mu]E \\ v_2 = -\alpha E + [\delta_2 + \theta + \eta]H \\ v_3 = -\eta H + \delta_3 B \end{bmatrix} \tag{35}$$

$$f = \begin{bmatrix} \beta \left(\frac{n_1 - n}{n_1}\right) L_m E A \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} [\varphi + \alpha + \delta_1 + \mu]E \\ -\alpha E + [\delta_2 + \theta + \eta]H \\ -\eta H + \delta_3 B \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial B} \\ \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial B} \\ \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial B} \end{bmatrix} \tag{36}$$

$$V = \begin{bmatrix} \frac{\partial v_1}{\partial E} & \frac{\partial v_1}{\partial H} & \frac{\partial v_1}{\partial B} \\ \frac{\partial v_2}{\partial E} & \frac{\partial v_2}{\partial H} & \frac{\partial v_2}{\partial B} \\ \frac{\partial v_3}{\partial E} & \frac{\partial v_3}{\partial H} & \frac{\partial v_3}{\partial B} \end{bmatrix} \tag{37}$$

Therefore, we obtain:

$$F = \begin{bmatrix} \beta \left(\frac{n_1 - n}{n_1}\right) L_m A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{38}$$

Let $[\varphi + \alpha + \delta_1 + \mu] = b_1$, $[\delta_2 + \theta + \eta] = b_2$

$$V = \begin{bmatrix} b_1 & 0 & 0 \\ -\alpha & b_2 & 0 \\ 0 & \eta & -\delta_3 \end{bmatrix} \tag{39}$$

Next, we evaluate F at ALFE

Hence,

$$F = \begin{bmatrix} \beta\left(\frac{n_1 - n}{n_1}\right) L_m \left(\frac{\Lambda}{\mu}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (40)$$

We then compute V^{-1}, using **Maple 2021**

$$V^{-1} = \begin{bmatrix} \frac{1}{b_1} & 0 & 0 \\ \frac{\alpha}{b_1 b_2} & \frac{1}{b_2} & 0 \\ \frac{\alpha\eta}{b_1 b_2 \delta_3} & -\frac{\eta}{b_2\delta_3} & -\frac{1}{\delta_3} \end{bmatrix} \quad (41)$$

We then compute FV^{-1}, which gives us the R_0

Therefore, $R_0 = \frac{\beta\, C_1 L_m C_2}{b_1}$. This is the basic Reproductive number for our general ALP model.

Thus, the basic reproduction number for the general ALP model is given by: $R_0 = \frac{\beta\, C_1 L_m C_2}{b_1}$.

$$R_0 = \frac{\beta(n_1 - n) L_m \Lambda}{n_1\ [\varphi + \alpha + \delta_1 + \mu\ ]\ \mu} \quad (42)$$

3.1.5 Local Stability Analysis of Algae-Lipid Free Equilibrium (ALFE) State of the General Model

This section examines local stability because it provides insight into the underlying dynamics of the algae lipid production (ALP) model and the likely effectiveness of control measures near the equilibrium state defined by equations (1) to (5).

Local stability analysis of the algae lipid-free equilibrium point T^0 is obtained by the variational matrix of the model (1) to (5) corresponding to the point T^0

Recall that, at equilibrium, the rate of change of each state variable is zero. The system at equilibrium is therefore written as:

$$x_1 = \frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A$$

$$x_2 = \frac{dE}{dt} = \lambda A - [\varphi + \alpha + \delta_1 + \mu\ ]E$$

$$x_3 = \frac{dH}{dt} = \alpha E - [\delta_2 + \theta + \eta]H$$

$$x_4 = \frac{dB}{dt} = \eta H - \delta_3 B$$

$$x_5 = \frac{dG}{dt} = \varphi E + \gamma G - [\epsilon + k + \mu]G$$

Recall: $\lambda = \beta\left(\frac{n_1 - n}{n_1}\right) L_m E$

is the force of algae lipid production for the algae population.

We then linearize equations (1) to (5) around the algae-lipid-free equilibrium (ALFE) by constructing the Jacobian matrix.

Considering the model (1) to (5) and by finding the partial derivatives of each equation with respect to the state variables, to present the following Jacobian matrix, J:

$$J(A, E, H, B, G) = \begin{pmatrix} \frac{\partial x_1}{\partial A} & \frac{\partial x_1}{\partial E} & \frac{\partial x_1}{\partial H} & \frac{\partial x_1}{\partial B} & \frac{\partial x_1}{\partial G} \\ \frac{\partial x_2}{\partial A} & \frac{\partial x_2}{\partial E} & \frac{\partial x_2}{\partial H} & \frac{\partial x_2}{\partial B} & \frac{\partial x_2}{\partial G} \\ \frac{\partial x_3}{\partial A} & \frac{\partial x_3}{\partial E} & \frac{\partial x_3}{\partial H} & \frac{\partial x_3}{\partial B} & \frac{\partial x_3}{\partial G} \\ \frac{\partial x_4}{\partial A} & \frac{\partial x_4}{\partial E} & \frac{\partial x_4}{\partial H} & \frac{\partial x_4}{\partial B} & \frac{\partial x_4}{\partial G} \\ \frac{\partial x_5}{\partial A} & \frac{\partial x_5}{\partial E} & \frac{\partial x_5}{\partial H} & \frac{\partial x_5}{\partial B} & \frac{\partial x_5}{\partial G} \end{pmatrix} \quad (43)$$

Solving the Jacobian obtained from (43) gives the required local stability conditions.

Therefore, since all our parameters $((\gamma, \mu, \epsilon, k, n_1, n, \eta, \delta_1, \delta_2, \delta_3, \theta, \alpha, \beta, \varphi, \Lambda) > 0)$

We therefore conclude that the algae-lipid-free equilibrium (ALFE) of the general system is locally asymptotically stable. Mathematically, this means that under the stated conditions the system returns to the lipid-free state after small perturbations.

3.1.6 Optimal Control Analysis

A further motivation for this work is to investigate optimal control, that is, to determine how strongly control measures can be applied to improve lipid production outcomes. In this study, the optimal strategy is one that

increases exposure in a controlled way, thereby enhancing harvesting and maximizing biofuel conversion (Breda et al., 2012).
In our Model (1) to (5), the rate at which the algae class (A) becomes ready for harvesting is determined by the mass action law (with transmission constant β) and harvesting class (H) converted at a constant per capita rate (η) and the rate of introducing new genetically modified algae (γ) into the (G) class. The motivation for using such a simple model is that it allows us to isolate the mechanisms involved in algae lipid production and to measure how systematically changing the algae lipid production assumptions changes the outcome.
Recall equations (1) to (5), which define the general model:
The deterministic compartmental model for algae lipid production is therefore represented by the following system of ordinary differential equations:

$$\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A$$
$$\frac{dE}{dt} = \lambda A - [\alpha + \delta_1 + \mu]E$$
$$\frac{dH}{dt} = \alpha E - [\delta_2 + \theta + \eta]H$$
$$\frac{dB}{dt} = \eta H - \delta_3 B$$
$$\frac{dG}{dt} = \gamma G - [\epsilon + k + \mu]G$$

Where $\lambda = \beta\left(1 - \frac{n(t)}{n_1}\right) E(t)L_m$ is the force of lipid accumulation.

3.1.7 Optimal Control Problem for the Algae Lipid Production Model

To optimize the bioprocess, we introduce three time-dependent control variables:
u_1: The effort applied to harvest lipid-rich algae (the compartment of H).
u_2: The effort applied to induce nutrient stress to trigger lipid accumulation (affecting the compartment of E).
u_3: The effort applied to introduce genetically modified algae with high lipid yield (compartment of G).
With these control variables in place, the controlled system is reformulated as follows:

$$\frac{dA}{dt} = \Lambda + kG + \theta H - \lambda A - \mu A \qquad (44)$$
$$\frac{dE}{dt} = \lambda A - [\alpha\, u_2 + \delta_1 + \mu]E \qquad (45)$$
$$\frac{dH}{dt} = \alpha u_2 E - [\delta_2 + \theta + \eta u_1]H \qquad (46)$$
$$\frac{dB}{dt} = \eta u_1 H - \delta_3 B \qquad (47)$$
$$\frac{dG}{dt} = \gamma u_3 G - [\epsilon + k + \mu]G \qquad (48)$$

with the constraints $0 \le u_i \le 1$ for $i = 1, 2, 3$
As in other real-world production systems, implementation costs must be considered alongside performance. For equations (44) to (48), these costs include the cost of harvesting algae, inducing nutrient stress to promote lipid accumulation, and introducing genetically modified high-lipid strains. This leads to the following optimal control objective function:

$$J(u_1, u_2, u_3) = \int_0^T \left[B(t) - \frac{1}{2}\left(C_1u_1(t) + C_2u_2(t) + C_3u_3(t)\right)\right] dt \qquad (49)$$

Because the aim of this analysis is to maximize the objective function, it can be interpreted in practical terms as maximizing economic return.
Accordingly, equation (49) can be written in the following business-oriented form:

$$Profit = Revenue - Costs \qquad (50)$$
$$J(u_1, u_2, u_3) \equiv Profit$$
$$\int_0^T \left[B(t) - \frac{1}{2}\left(C_1u_1(t) + C_2u_2(t) + C_3u_3(t)\right)\right] dt \equiv Revenue - Cost$$

This revenue is also seen as the instantaneous benefit of the process, defined as the quantity of biofuel produced at time, t. Its integral $\int_0^T B(t)\, dt$ gives the total biofuel yield, which is the primary product to be maximized.
Here, our goal is to find the optimal control functions u_1^*, u_2^*, u_3^* such that
$J(u_1^*, u_2^*, u_3^*) = \max_{u_1,u_2,u_3} \{J(u_1,\ u_2,\ u_3)\}$
We now apply Pontryagin's maximum principle and solve the resulting system.
This yields the final practical characterization of the optimal control values:
The compact form of the optimal control characterization is given by:

$$\begin{cases} u_1^* = \min\left\{\max\left\{0, \frac{(\lambda_4-\lambda_3)\,\eta \mathrm{H}_a}{C_1}\right\}, 1\right\} \\ u_2^* = \min\left\{\max\left\{0, \frac{(\lambda_3-\lambda_2)\alpha E}{C_2}\right\}, 1\right\} \\ u_3^* = \min\left\{\max\left\{0, \frac{\lambda_5 \gamma G}{C_3}\right\}, 1\right\} \end{cases} \qquad (51)$$

3.1.8 Concave Test Analysis

Because the controls are separable, we examine the second partial derivatives with respect to each control variable:
For u_1:

$$\frac{\partial H}{\partial u_1} = -C_1 u_1 + (\lambda_4 - \lambda_3)\,\eta \mathrm{H}_a$$

$$\frac{\partial^2 H}{\partial u_1^2} = -C_1 \qquad (52)$$

For u_2:

$$\frac{\partial H}{\partial u_2} = -C_2 u_2 + (\lambda_3 - \lambda_2)\,\alpha\, E$$

$$\frac{\partial^2 H}{\partial u_2^2} = -C_2 \qquad (53)$$

For u_3:

$$\frac{\partial H}{\partial u_3} = -C_3 u_3 + \lambda_5 \gamma G$$

$$\frac{\partial^2 H}{\partial u_3^2} = -C_3 \qquad (54)$$

The cost C_1, C_2, C_3 coefficients are strictly positive, as they represent weights on the cost of implementing the controls. Therefore, the second derivatives of the Hamiltonian with respect to the control variables are:

$$\frac{\partial^2 H}{\partial u_1^2} = -C_1 < 0,\ \frac{\partial^2 H}{\partial u_2^2} = -C_2 < 0, \frac{\partial^2 H}{\partial u_3^2} = -C_3 < 0$$

Since the second derivatives with respect to the control variables are negative, it implies that the Hamiltonian is strictly concave in each control variable u_i and the formal analysis confirms that the model is well-posed for optimization.

3.1.9 Sensitivity Analysis of the General Algae Lipid Production (ALP) Model with respect to R_0

This section examines how changes in model parameters influence algae lipid extraction and conversion. To do so, we perform a sensitivity analysis of the algae lipid production (ALP) model and evaluate how variation in individual parameters affects the model output, here represented by R_0.

This Analysis Helps Identify Whether Increasing a Given Parameter Will Raise or Reduce the Model Output, And Therefore Highlights the Parameters with The Greatest Practical Importance.

We Use the Normalized Forward Sensitivity Index of a Variable That Depends On a Parameter, Expressed as:

$$\mathrm{r}_p^v = \frac{\partial v}{\partial p} \times \frac{p}{v}$$

In particular, we compute the sensitivity indices of the basic reproduction number R_0 with respect to the parameters of the model.

A positive sensitivity index means that increasing a parameter increases R_0, whereas a negative index means that increasing the parameter decreases R_0. This information is useful for identifying which parameters most strongly promote or suppress lipid production in the model defined by equations (1) to (5).

Recall that

$$R_0 = \frac{\beta\, C_1 L_m C_2}{b_1} = \frac{\beta (n_1 - n) L_m \Lambda\ [\varphi + \alpha + \delta_1 + \mu]}{\mu\, n_1},$$

Hence, using **Maple 2021**, to compute the sensitivity index of a parameter, say, β with respect to R_0 as:

$$r_{\beta}^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0}$$

$$= \frac{\partial\left(\frac{\beta(n_1 - n)L_m\Lambda\ [\varphi + \alpha + \delta_1 + \mu]}{\mu\, n_1}\right)}{\partial \beta} \times \frac{\beta}{\left(\frac{\beta(n_1 - n)L_m\Lambda\ [\varphi + \alpha + \delta_1 + \mu]}{\mu\, n_1}\right)}$$

$$= \left(\frac{\beta\mu\, n_1}{\beta(n_1 - n)L_m\Lambda\ [\varphi + \alpha + \delta_1 + \mu]}\right) \times \frac{\beta(n_1 - n)L_m\Lambda\ [\varphi + \alpha + \delta_1 + \mu]}{\beta\mu\, n_1} = 1(positive) \qquad (55)$$

We then compute the sensitivity index of the remaining parameters using the same approach as that of (β), to give us the table below:

Table 3: Sensitivity index of R_0 with respect to parameters of the model

Parameters	Numerical value	Sensitivity index
μ	0.0000421	-0.998201
η	0.0001	1
β	0.0002	1
γ_1	0.58	-0.3365
Λ	20	1
δ_1	0.000001	-0.5274
α	0.0001	-0.0007234
θ	0.76	-1

Table 3 provides sensitivity index results that clarify how parameter changes influence the response variable of the algae lipid production model. In other words, it shows how strongly each parameter affects the basic reproduction number R_0. A positive sensitivity index indicates a direct relationship between a parameter and the model's basic reproduction number. Increasing such a parameter increases R_0, while decreasing it lowers R_0. By contrast, a negative sensitivity index indicates an inverse relationship: increasing the parameter reduces R_0, whereas decreasing it raises R_0. Overall, the results suggest that the most influential parameters should receive priority when designing strategies to regulate algae lipid production and improve biofuel output.

3.2 Model Simulation and Explanation

To validate the mathematical representation of algae lipid production dynamics, we varied key parameters and examined how these changes affected lipid output over time.

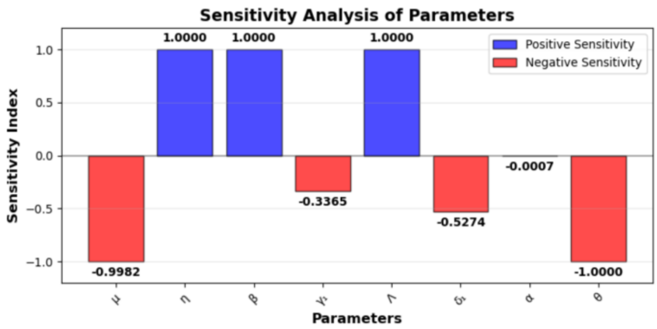

Figure 2: Local Sensitivity index analysis of the R_0 for the proposed compartmental model.

Figure 2 presents a bar plot graph of the sensitivity index, illustrating how small changes in different model parameters can help predict their impact on the average lipid production output (R_0) in algae. The blue represents positive bar plots, which indicate that increasing these positive parameters will increase lipid production. In contrast, the negative bar plots are given a red color, meaning that increasing this parameter will decrease lipid production, and vice versa.

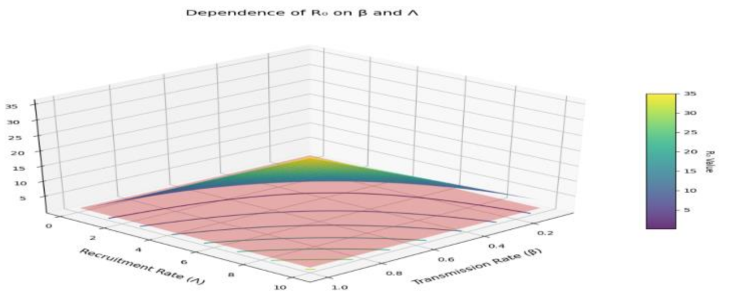

Figure 3:Three-dimensional response surface of R_0 as a function of Λ , $and\ \beta$

Figure 3 illustrates how the parameters Λ and β influence R_0. The parameter β represents the lipid transmission or production rate, while Λ represents the recruitment of new algae into the production system. A reduction in either parameter lowers overall lipid production and may eventually halt sustained production.

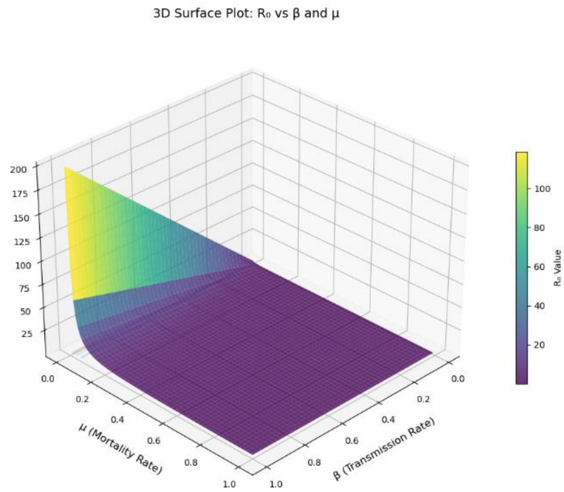

Figure 4: Three-dimensional response surface of R_0 as a function of $\mu\ and\ \beta$

Figure 4 shows the relationship among μ, β, and R_0 in the algae lipid production model. The plot suggests that optimizing growth conditions that improve β has a stronger positive effect on lipid production than merely reducing mortality μ, although keeping algae healthy remains important.

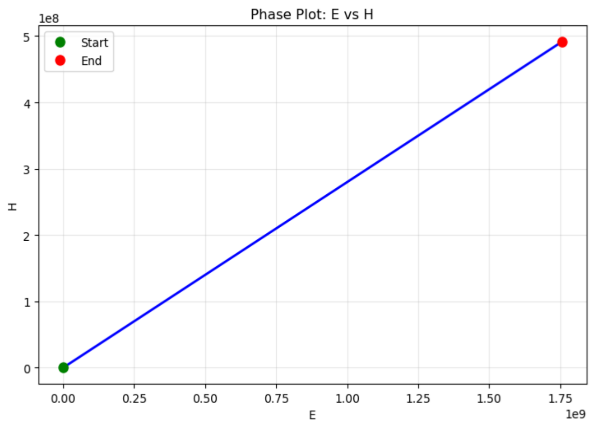

Figure 5: state-space phase diagram of the algal lipid production system in the (H,E) plane, depicting the nonlinear coupling between the retained stress-adapted population (E) and the harvested output stream (H).

Figure 5 highlights the trade-off involved in harvesting. Harvesting too little leaves valuable biomass underused, whereas harvesting too aggressively can damage the standing algae population and reduce future yield. The figure therefore points to an equilibrium level that balances immediate recovery with long-term sustainability.

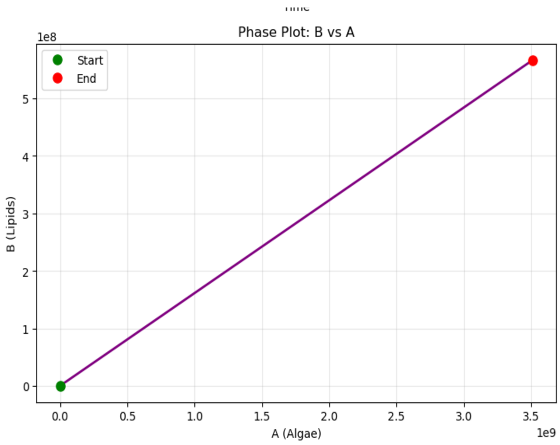

Figure 6: state-space representation of the biomass-to- biofuel conversion kinetics in the (A,B) plane.

Figure 6 illustrates the relationship between lipid accumulation and algae population size, emphasizing how production output depends on maintaining a viable biomass base.

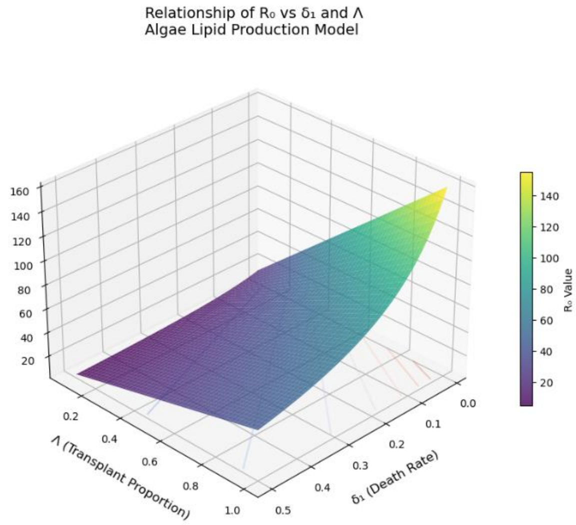

Figure 7: 3D plot showing the relationship between δ_1, Λ and R_0

From Figure 7, we have a plot visualizing the relationship between δ_1, Λ and R_0 in our algae lipid production dynamics model. This graph is a powerful tool that shows you the best ways to manage your algae farm to ensure it is productive and successful.

4.0 Conclusion

This study developed a computational framework that captures the coupled dynamics of algae growth, lipid accumulation, harvesting, and conversion for sustainable aviation fuel production. The model provides a useful way to understand the nonlinear interactions that shape production performance. The derivation of the basic reproduction number R_0 is one of the central outcomes of the study because it offers a compact measure of system viability and production efficiency. Overall, the results indicate that large-scale algae lipid production for SAF is theoretically feasible, but its success depends strongly on environmental stressors, mortality effects, and operational control.

Conflict of Interest

The authors declare no conflict of interest.

5.0 Recommendation

- Boost β, Λ, η – Maximize contact, recruitment, and conversion rates.
- Reduce mortality (μ, δ_1) and sporulation (θ) via monitoring.
- Apply optimal controls – Cycle stress, pulse harvest, phase GM strains.
- Validate with pilot data – Calibrate model parameters experimentally.
- Deploy community pilots in Nigeria – Use wastewater, track $R_0 > 1$, commercialize local strains.

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Nomenclature

$A(t)$:	Algae population.
$E(t)$:	Algae exposed to environmental factors, accumulating more lipids.
$H(t)$:	Algae population available for harvesting.
$B(t)$:	Algae population designated for biofuel production.
η:	Conversion rate.
λ:	Lipid accumulation force due to environmental factors.
μ:	Natural algae death rate.
θ:	Spore formation rate.
δ_1:	Algae death rate due to environmental factors.
β_2:	Algae contact rate.
δ_2:	Harvest rate.
δ_3:	Biofuel conversion rate.
α:	Lipid accumulation rate due to exposure.
Λ:	Algae purchased or planting proportion.
β:	Lipid accumulation efficiency.
$x(t)$:	Biomass concentration.
k:	Carrying capacity.
μ:	natural death rate.

Declaration of Generative AI and AI-assisted technologies in the writing process
During the preparation of this manuscript, the authors used DeepSeek to assist with language refinement, grammar correction, and clarity enhancement of the text. The tool was employed solely to improve readability and ensure consistent scientific terminology. All intellectual content, including the research design, mathematical modelling, data analysis, interpretation, and final conclusions, remains the original work of the authors. The use of AI did not influence the scientific integrity or originality of this research. After using DeepSeek, the authors reviewed and edited the content as needed and take full responsibility for the final manuscript

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